MODELS FOR COMMUTING CONTRACTIONS

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1. INTRODUCTION

Let L denote the unilateral shift on a vector-valued H^2 space $H^2_{\mathscr{H}}$. Let M be a closed invariant subspace for L, let P project $H^2_{\mathscr{H}}$ onto $M^{\perp}$, and denote by T the restricted shift $T = PL\,|_{M^{\perp}}$. G.-C. Rota showed by an amazingly simple argument that T^* is a universal model for a large class of operators; that is, if S is a contraction on $\mathscr{H}$ of norm less than 1 then S is similar to some T^*. Rota's technique was refined by L. de Branges and J. Rovnyak to yield a model up to unitary equivalence: S is unitarily equivalent to some T^* if and only if $\|S\| \leq 1$ and $S^n \to 0$ strongly. (Both these results and basic background material on shifts and vectorial function theory can be found in [3].) Using restricted shifts as models, an extensive structure theory for operators has been developed; see [10].

D. N. Clark [1] extended Rota's theorem to the case of N commuting contractions by using the maps $T_k = PL_k\,|_{M^{\perp}}$ as a model, where

$$L_k f(z_1, \cdots, z_N) = z_k f(z_1, \cdots, z_N)$$

in the polydisc space $H^2(U^N)$. (See [6] for a basic reference.) Clark also characterized the commutant of $\{T_1, \cdots, T_N\}$, and hence, up to similarity, the commutant of N commuting contractions; this extended the one variable results of D. Sarason [7] and B. Sz.-Nagy and C. Foiaş [10]. In this paper, we modify the de Branges-Rovnyak technique to construct a unitary equivalence model for N commuting contractions by using a weighted shift analog of the maps T_k, and we extend Clark's description of the commutant to this case. We explain our notation below; for the basic theory of one variable weighted shifts the reader can consult [8].

2. NOTATION AND MODELS

For a fixed positive integer N, we use the notation $z = (z_1, \cdots, z_N)$, $e^{i\phi} = (e^{i\phi_1}, \cdots, e^{i\phi_N})$, and $J = (j_1, \cdots, j_N)$ a multi-index of either nonnegative integers, which we indicate by $J \geq 0$, or arbitrary integers. We let e_k denote the multi-index J with $j_k = 1$ and $j_n = 0$ otherwise; $J \pm e_k$ has the obvious meaning except that by using $J - e_k$ we imply that $j_k \geq 1$. We use $J \cdot \phi = j_1\phi_1 + \cdots + j_N\phi_N$; for $J \geq 0$, $|J| = j_1 + \cdots + j_N$, $J! = j_1! \cdots j_N!$, $z^J = z_1^{j_1} \cdots z_N^{j_N}$; and given commuting operators $S_1, \cdots, S_N$, $S^J = S_1^{j_1} \cdots S_N^{j_N}$. We let U^N and T^N denote the N-dimensional unit polydisc and torus respectively.

For a separable Hilbert space $\mathscr{H}$, $L^2 = L^2_{\mathscr{H}}(T^N)$ and $H^2 = H^2_{\mathscr{H}}(U^n)$ denote the standard Lebesgue and Hardy spaces of square summable vector-valued functions from T^N into $\mathscr{H}$: $f \in L^2$ has Fourier expansion

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