AN ANALYTIC CHARACTERIZATION OF GEOMETRICALLY STARLIKE FUNCTIONS

David Styer

The class $S^*(\alpha)$ of functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic and univalent in the unit disc $\mathbb{B}$, and which satisfy the condition $|\arg zf'(z)/f(z)| < \alpha\pi/2$, $0 < \alpha \leq 1$, was introduced and studied by Brannan and Kirwan [1]. Since $S^*(1)$ is the usual class of starlike univalent functions, $S^*(1)$ is not studied as such. Recently Leach [3] has extended $S^*(\alpha)$ to multivalent starlike functions. A slight modification of Leach's definition (which yields the same class of functions) follows.

Let p be a positive integer, and $0 \leq \alpha \leq 1$. Then $S^*(\alpha, p)$ denotes those functions f holomorphic in $\mathbb{B}$, with exactly p zeros there (zeros and critical points are counted by their multiplicity), such that

$$\limsup_{r \to 1^-} \max_{|z| = r} |\arg zf'(z)/f(z)| \leq \alpha\pi/2 .$$

This definition clearly has affinities with the analytic definition of weakly starlike functions, $S_w(p)$, as defined by Hummel [2]. $f \in S_w(p)$ if and only if f is holomorphic in $\mathbb{B}$, has exactly p zeros there, and

$$\liminf_{r \to 1^-} \min_{|z| = r} \Re\, zf'(z)/f(z) \geq 0 .$$

For $p > 1$ it is no longer true that, with $\alpha = 1$, $S^*(\alpha, p) = S_w(p)$. It is the purpose of this paper to prove

THEOREM 1. $S^*(1, p) = S_g(p)$.

Here $S_g(p)$ is the class of geometrically starlike functions of order p. That is, $f \in S_g(p)$ if and only if f is holomorphic in $\mathbb{B}$, has exactly p zeros there, and for each point z in $\mathbb{B}$ there is a curve in $\mathbb{B}$, between z and some zero of f, which f maps one-to-one onto the radial line segment between 0 and f(z). An interesting feature of Theorem 1 is that the proof is almost completely combinatorial. Setting the tone, we will need

LEMMA 1. *Let* $f \in S_w(p)$. *Then* $f \in S_g(p)$ *if and only if* f *has* p - 1 *critical points.*

This result may be found in Styer [4, p. 232].

For all $r > 0$ define $\gamma_r\colon [0, 2\pi] \to \mathbb{C}$ by $\gamma_r(t) = re^{it}$, and let C_r be the image of γ_r. For any closed curve γ which does not pass through 0, let $I(\gamma, 0)$ be the index, or winding number, of γ about 0.

LEMMA 2. *Let* g *be holomorphic in the annulus* $\{z : 0 < \rho < |z| < 1\}$, *and have no zero there. Then the following two statements are equivalent.*

Received October 15, 1975.

Michigan Math. J. 23 (1976).