ON AUTOMORPHIC FORMS AND CARLESON SETS

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1. INTRODUCTION

Let \( \Gamma \) be a Fuchsian group in the unit disk \( D \subset \mathbb{C} \) and let \( L \subset \partial D \) be its limit set. An automorphic form of weight \( q \) (\( q = 0, \pm 1, \cdots \)) is an analytic function \( f(z) \) (\( z \in D \)) such that

\[
f(\gamma(z)) \gamma'(z)^q \equiv f(z) \quad (\gamma \in \Gamma).
\]

Let \( A_2^\infty(\Gamma) \) be the space of automorphic forms of weight 2 with

\[
\sup_{z \in D} (1 - |z|^2)^2 |f(z)| < \infty.
\]

This space was introduced by L. Bers [1] and has applications, for instance, in Teichmüller space theory [2, p. 272]. The theory of the related spaces \( A_p^\infty(\Gamma) \) (\( 1 \leq p \leq \infty, q = 2, 3, \cdots \)) is described, for instance, in the book of Kra [5].

The Eichler integral of \( f \in A_2^\infty(\Gamma) \) is defined by

\[
h(z) = \frac{1}{2} \int_0^z (\xi - z)^2 f(\xi) \, d\xi \quad (z \in D);
\]

that is, by \( h''(z) = f(z) \) and \( h(0) = h'(0) = h''(0) = 0 \). It follows from (1.1) that

\[
h(\gamma(z))/\gamma'(z) = h(z) + c_\gamma(z) \quad (\gamma \in \Gamma),
\]

where the Eichler period \( c_\gamma(z) \) is a polynomial of degree \( \leq 2 \). The Eichler periods are elements of the Eichler cohomology group \( H^1(\Gamma, \Pi_2) \) [5, pp. 148, 196], and (1.4) defines a homomorphism from \( A_2^\infty(\Gamma) \) into \( H^1(\Gamma, \Pi_2) \). Bers [1] has shown that this homomorphism is injective for groups of the first kind (that is, \( L = \partial D \)). We shall prove that it is injective if and only if \( L \) is not a Carleson set.

A closed set \( E \subset \partial D \) is called a Carleson set if

\[
\sum_n \ell_n = 2\pi, \quad \sum_n \ell_n \log \frac{2\pi}{\ell_n} < \infty,
\]

where \( \ell_n \) are the lengths of the component arcs of \( \partial D \setminus E \). It was proved by L. Carleson [3] that if a function is analytic in \( D \) and belongs to \( \text{Lip}\alpha \) for some \( \alpha > 0 \), then its zero set on \( \partial D \) is a Carleson set; conversely, every Carleson set is the zero set on \( \partial D \) of an analytic function even with bounded derivative. We shall use results of Taylor and Williams [9] and of Nelson [7] on Carleson sets.

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