ON AUTOMORPHIC FORMS AND CARLESON SETS

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1. INTRODUCTION

Let Γ be a Fuchsian group in the unit disk $D \subset \mathbb{C}$ and let $L \subset \partial D$ be its limit set. An automorphic form of weight q $(q = 0, \pm 1, \cdots)$ is an analytic function $f(z)$ $(z \in D)$ such that

$$(1.1) \qquad f(\gamma(z))\,\gamma'(z)^q \equiv f(z) \qquad (\gamma \in \Gamma).$$

Let $A_2^\infty(\Gamma)$ be the space of automorphic forms of weight 2 with

$$(1.2) \qquad \sup_{z \in D}\,(1 - |z|^2)^2\,|f(z)| < \infty.$$

This space was introduced by L. Bers [1] and has applications, for instance, in Teichmüller space theory [2, p. 272]. The theory of the related spaces $A_q^p(\Gamma)$ $(1 \leq p \leq \infty,\ q = 2, 3, \cdots)$ is described, for instance, in the book of Kra [5].

The *Eichler integral* of $f \in A_2^\infty(\Gamma)$ is defined by

$$(1.3) \qquad h(z) = \frac{1}{2}\int_0^z (\zeta - z)^2 f(\zeta)\,d\zeta \qquad (z \in D);$$

that is, by $h'''(z) = f(z)$ and $h(0) = h'(0) = h''(0) = 0$. It follows from (1.1) that

$$(1.4) \qquad h(\gamma(z))/\gamma'(z) = h(z) + c_\gamma(z) \qquad (\gamma \in \Gamma),$$

where the *Eichler period* $c_\gamma(z)$ is a polynomial of degree ≤ 2. The Eichler periods are elements of the Eichler cohomology group $H^1(\Gamma, \Pi_2)$ [5, pp. 148, 196], and (1.4) defines a homomorphism from $A_2^\infty(\Gamma)$ into $H^1(\Gamma, \Pi_2)$. Bers [1] has shown that this homomorphism is injective for groups of the first kind (that is, $L = \partial D$). We shall prove that it is injective if and only if L is not a Carleson set.

A closed set $E \subset \partial D$ is called a *Carleson set* if

$$(1.5) \qquad \sum_n \ell_n = 2\pi, \qquad \sum_n \ell_n \log\frac{2\pi}{\ell_n} < \infty,$$

where ℓ_n are the lengths of the component arcs of $\partial D \setminus E$. It was proved by L. Carleson [3] that if a function is analytic in D and belongs to $\mathrm{Lip}\,\alpha$ for some $\alpha > 0$, then its zero set on ∂D is a Carleson set; conversely, every Carleson set is the zero set on ∂D of an analytic function even with bounded derivative. We shall use results of Taylor and Williams [9] and of Nelson [7] on Carleson sets.

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