LACUNARY POWER SERIES ON THE UNIT CIRCLE

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By the statement that a formal power series

(1) $$S(\theta) = \sum_{n=1}^{\infty} c_n e^{ik_n\theta}$$

is q-*lacunary* we shall mean that its exponents k_n satisfy a condition of the form $k_{n+1}/k_n > q > 1$ $(n = 1, 2, \cdots)$. In a research announcement [2], R.E.A.C. Paley stated that *if the series* (1) *is* q-*lacunary, and if in addition* $|c_n| \to 0$ *and* $\sum |c_n| = \infty$, *then for each finite complex number* w *the series converges to* w *at every point of a set that is dense in* $[0, 2\pi]$.

A complete proof of Paley's theorem was later given by M. Weiss [3]. Subsequently, J.-P. Kahane, M. Weiss, and G. Weiss [1, pp. 1-16] showed that the plane-covering property of $S(\theta)$ is only one aspect of a much stronger property of the sequence $\{S_n\}$ of partial sums of (1). They proved that *if the series* (1) *is* q-*lacunary, and if in addition* $c_n \to 0$ *and* $\sum |c_n| = \infty$, *then corresponding to every closed connected subset* C *of the extended complex plane there exists an everywhere dense set* E *in* $[0, 2\pi]$ *such that for each* θ *in* E *the set* C *is the set of limit points of* $\{S_n(\theta)\}$.

This theorem fails if we omit the hypothesis that $c_n \to 0$. Indeed, let

$$E(\infty, S) = \{\theta \in [0, 2\pi] : \lim_{n\to\infty} |S_n(\theta)| = \infty\}.$$

If for each index n we take $c_n = n!$, then (even without the hypothesis of lacunarity) the series (1) obviously has the property that $E(\infty, S) = [0, 2\pi]$. It is not known in general whether the set $E(\infty, S)$ remains dense in $[0, 2\pi]$ if $c_n \not\to 0$. Simple arguments show that it is a dense set if we assume in addition that $q > 3$. In this note we prove the following result.

THEOREM. *To each* $q > 1$ *there corresponds a positive constant* A_q *such that for each* q-*lacunary series* (1) *satisfying the two conditions*

$$\limsup_{n\to\infty} |c_n| > 0$$

and

(2) $$\liminf_{N\to\infty} \left(\sum_{n=1}^{N} |c_n| \Big/ \max_{1\leq n\leq N} |c_n| \right) > A_q,$$

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