TRANSLATION-INVARIANT OPERATORS ON $L^p(G)$, $0 < p < 1$

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Let G be a compact abelian group, and for $0 < p \leq \infty$ let $L^p(G)$ denote the usual Lebesgue space with respect to normalized Haar measure on G. For $g \in G$ and functions f on G we define the translation operator T_g by $T_g f(h) = f(h - g)$ for $h \in G$. The collection $\{T_g: g \in G\}$ is a group of linear isometries on any $L^p(G)$, and we are interested in the bounded linear operators on $L^p(G)$ which commute with this group—the translation-invariant linear operators on $L^p(G)$. The problem of characterizing these operators is sometimes known as the multiplier problem and, for $p \geq 1$, has attracted much attention. Satisfactory characterizations are available only for the case $p = 1$ and the trivial case $p = 2$. Obtaining such a characterization for any other $p \geq 1$ appears to be a most difficult task, but for $p < 1$ the problem seems to have been neglected. The purpose of this note is to present such a characterization when $0 < p < 1$.

THEOREM. *Let* G *be a compact abelian group and fix* p *with* $0 < p < 1$. *The bounded linear operators on* $L^p(G)$ *which commute with each* T_g $(g \in G)$ *are precisely those operators of the form*

$$\sum_{i=1}^{\infty} a_i T_{g_i}, \quad \text{where } g_i \in G \text{ and } \sum_{i=1}^{\infty} |a_i|^p < \infty. \tag{1}$$

Proof. It is obvious that (1) defines a bounded and translation-invariant operator on $L^p(G)$. To show that each such operator is of the form (1), we require two lemmas.

LEMMA 1. *Let* K *be a compact Hausdorff space and let* λ *be a complex-valued regular Borel measure on* K. *If for some* p $(0 < p < 1)$ *and some finite positive number* M *we have*

$$\sum_{j=1}^{m} |\lambda(E_j)|^p \leq M \tag{2}$$

for each m *and each finite Borel partition* $\{E_j\}_{j=1}^m$ *of* K, *then* λ *is of the form* $\sum_{i=1}^{\infty} a_i \delta_{x_i}$, *where* δ_{x_i} *is the unit mass at some point* $x_i \in K$ *and* $\sum_{i=1}^{\infty} |a_i|^p \leq M$.

Proof. Assume first that λ is positive and let $\lambda = \lambda_d + \lambda_c$ be the decomposition of λ into discrete and continuous parts. Then (2) holds with either λ_d or λ_c in place of λ. If $\lambda_c(K) > 0$, then, as a consequence of [1, 11.44], for any $m = 1, 2, \cdots$ we can find disjoint Borel subsets $E_1, \cdots, E_m$ of K such that $\lambda_c(E_j) = m^{-1}\lambda_c(K)$, $j = 1, \cdots, m$. For these E_j we have

$$\sum_{j=1}^{m} |\lambda_c(E_j)|^p = m(m^{-1}\lambda_c(K))^p = m^{1-p}(\lambda_c(K))^p,$$

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