ON INNER FUNCTIONS WITH B^p DERIVATIVE

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As the title suggests, this paper contains results similar to those in [2], with the spaces H^p replaced by B^p. The basic problem we consider is that of determining the B^p classes $(p > 0)$ to which the derivative ϕ' of an inner function ϕ in the unit disk belongs. Recall that the space B^p is by definition the class of functions $f(z)$ analytic in the unit disk U and satisfying

$$\|f\|_p = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})| (1 - r)^{1/p-2} \, d\theta dr < \infty .$$

(Here and in what follows, $d\theta$ denotes *normalized* Lebesgue measure on the unit circle.)

M. R. Cullen [8] first considered the problem of determining the B^p classes of ϕ', for ϕ a singular inner function, and he conjectured that $\phi' \notin B^{1/2}$ for such a function. Cullen's idea was to use this to prove a conjecture of J. G. Caughran and A. L. Shields [6] to the effect that $\phi' \notin H^{1/2}$. H. A. Allen and C. L. Belna [3] disproved Cullen's conjecture by giving examples of singular inner functions ϕ with $\phi' \in B^p$ for all $p < 2/3$. The conjecture that $\phi' \notin B^{2/3}$ for inner functions with singular factors then seemed reasonable (see, for instance, Caughran and Shields [7]). Finally, D. Protas [11] gave a sufficient condition for $\phi' \in B^p$ $(p > 1/2)$ for ϕ a Blaschke product. (For $p < 1/2$, we have $\phi' \in B^p$ for any inner function [9, Theorem 5].)

In this paper we prove that if ϕ has a singular factor, then $\phi' \notin B^{2/3}$. To do this we develop (in Section 1) an integrated analogue of the angular derivative, the latter having been used in [2] to prove, among other things, the $H^{1/2}$ conjecture of Caughran and Shields. The methods of Section 1 are also applied to give a sufficient condition for the relation $\phi' \in B^p$ for ϕ a singular inner function (Section 3), to give a partial converse to Protas' condition for Blaschke products (Section 4), and to show that both Protas' condition and the partial converse are "best possible" (Sections 4 and 5).

The original $H^{1/2}$ conjecture of Caughran and Shields in [6] arose in connection with problems on exceptional sets, and the solution of the $B^{2/3}$ conjecture has applications to exceptional sets, as did our solution of the $H^{1/2}$ conjecture in [2]. These applications are discussed in Section 2.

Throughout this paper, the similarity of our results with those in [2] is apparent; however, it seems unlikely that the results of the present paper can be obtained directly from those in [2]. One reason for this is our example (Lemma 2) of a Blaschke product B with $B' \in B^{2/3}$ but $B' \notin H^{1/2}$.

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