THE SCHWARTZ-HILBERT VARIETY

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1. INTRODUCTION

Let θ be the collection of locally convex topological vector spaces (LCS) that are embeddable as subspaces of a power X^I, for each infinite-dimensional Banach space X. It is known that θ contains all nuclear spaces (S. A. Saxon [16]), that only Schwartz spaces are in θ (J. Diestel and R. H. Lohman [4]), and that there are Schwartz spaces not in θ (the author [1] and D. J. Randtke [15]). We shall show (Theorem 4.1) that θ coincides with the Schwartz-Hilbert variety (see below). Thus θ is strictly larger than the nuclear variety [1].

Our main tool (see the Lemma) is A. Dvoretsky's theorem [6] (see [12, p. 42]) on the existence of near-ℓ_2^n-subspaces in any Banach space. Using Dvoretsky's theorem, we show that each compact map into Hilbert space can be factored through a subspace of any infinite-dimensional Banach space (Theorem 3.3). For similar results, see C. P. Stegall and J. R. Retherford [18].

2. NOTATION AND PRELIMINARIES

X, Y, and Z are reserved for infinite-dimensional Banach spaces. We write ℓ_2 for the Hilbert space of squared-summable sequences. R, S, T, U, V are reserved for bounded linear maps. Each $T = T_\lambda$ represents a diagonal map on ℓ_2 (that is, $\lambda = (\lambda_n)$ and $T_\lambda(\alpha_n) = (\lambda_n \alpha_n)$). We note that T_λ is a positive compact map if and only if $\lambda_n \geq 0$ for all n and (λ_n) belongs to the space c_0 of null sequences. To say $S: X \to Y$ factors through Z means that there are maps $U: X \to Z$ and $V: Z \to Y$ such that $S = VU$.

A *prevariety* [2] is a collection of LCS's that is closed with respect to the formation of subspaces and arbitrary products. A *variety* [5] is a prevariety that, in addition, contains all its separated quotients. If X is a Banach space, we denote by $\rho\nu(X)$ (respectively $\nu(X)$) the smallest prevariety (variety) containing X.

It follows from Theorem 1.1 of [5, p. 209], that for each LCS E and each Banach space X, $E \in \rho\nu(X)$ if and only if E is a subspace of some power of X. A *universal generator* [5] for a variety $\mathscr{A}$ is an $E \in \mathscr{A}$ such that each $F \in \mathscr{A}$ is embeddable as a subspace of a power of E.

Let $\mathscr{S}$ be the variety of Schwartz spaces (see [8, p. 271]), let $\mathscr{H}$ be the variety $\nu(\ell_2)$, and let $\mathscr{S}\mathscr{H}$ be their intersection, the *Schwartz-Hilbert* variety. From Theorem 4.4 of [5, p. 219] and the definition of Schwartz spaces it follows that each $E \in \mathscr{S}\mathscr{H}$ has a neighborhood basis $\mathscr{U}$ such that the completion of the norm space E_U [17, p. 53] is a Hilbert space, for each $U \in \mathscr{U}$. Furthermore, for each $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$ such that the canonical map $E_V \to E_U$ [17, p. 53] is precompact. In the language of [14] and [15], each $E \in \mathscr{S}\mathscr{H}$ is a subspace of a compact projective limit of ℓ_2-spaces. Finally, let $\mathscr{N}$ be the variety of nuclear spaces (see [13]).

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