CONTRACTIVE LINEAR MAPS ON C*-ALGEBRAS

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0. INTRODUCTORY REMARKS

The purpose of this note is to study the interplay and distinctions between contractive and completely contractive linear maps on C^*-algebras. Both in spirit and in technique, these results follow the outline given by M.-D. Choi [2].

If $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras with identity, and $\phi: \mathscr{A} \to \mathscr{B}$ is a linear map, then $\phi_n = \phi \otimes \mathrm{id}_n$ is the entry-wise map from the C^*-algebra $\mathscr{A} \otimes M_n$ to $\mathscr{B} \otimes M_n$, where M_n denotes the C^*-algebra of n-by-n complex matrices. We say that ϕ is *completely positive* if every ϕ_n $(n \geq 1)$ is positive; ϕ is *completely contractive* if $\sup_n \|\phi_n\| \leq 1$; and ϕ is *completely bounded* if $\sup_n \|\phi_n\| < \infty$; see [1]. Note that $\|\phi_n\| \geq \|\phi\|$.

Let $C_k[\mathscr{A}, \mathscr{B}]$ denote the set of all linear maps ϕ from $\mathscr{A}$ to $\mathscr{B}$ such that $\phi_1, \cdots, \phi_k$ are contractive; $C_\infty[\mathscr{A}, \mathscr{B}]$ is then the set of all completely contractive maps. It is easy to see that $C_1 \supseteq C_2 \supseteq \cdots$ and $C_\infty = \bigcap_{k \geq 1} C_k$.

It is known that if $P_k[\mathscr{A}, \mathscr{B}]$ denotes the set of all linear maps ϕ from $\mathscr{A}$ to $\mathscr{B}$ such that $\phi_1, \cdots, \phi_k$ are positive, then $P_1[\mathscr{A}, \mathscr{B}] = P_\infty[\mathscr{A}, \mathscr{B}]$ if either $\mathscr{A}$ or $\mathscr{B}$ is commutative [1, p. 144]. Further, Choi established that $P_1[\mathscr{A}, \mathscr{B}] = P_2[\mathscr{A}, \mathscr{B}]$ implies $\mathscr{A}$ or $\mathscr{B}$ is commutative [2, Theorem 4].

The results we shall establish are analogous: if $\mathscr{B}$ is commutative, then $C_1[\mathscr{A}, \mathscr{B}] = C_\infty[\mathscr{A}, \mathscr{B}]$; and if $C_1[\mathscr{A}, \mathscr{B}] = C_2[\mathscr{A}, \mathscr{B}]$, then $\mathscr{A}$ or $\mathscr{B}$ is commutative. The analogy breaks down drastically in the case of a commutative domain: If $\mathscr{A}$ is commutative, therefore of the form C(X) [5, Theorem 4.2.2] and $C_1[\mathscr{A}, \mathscr{B}] = C_\infty[\mathscr{A}, \mathscr{B}]$, then by Theorem C, the space X contains at most two points! We shall also make some remarks about the case of completely bounded maps.

1. COMPLETELY CONTRACTIVE MAPS

LEMMA 1. *Let $\mathscr{S}$ be a linear subspace of a C^*-algebra, and let $\mathscr{C}$ be a commutative C^*-algebra. Let $\phi: \mathscr{S} \to \mathscr{C}$ be a linear map. Then $\|\phi\| = \|\phi_n\|$ for $n = 1, 2, \cdots$.*

Proof. We modify [1, Proposition 1.2.2]. Identify $\mathscr{C}$ as C(X), let n be a positive integer, and for $[a_{ij}] \in \mathscr{S} \otimes M_n$, let $\phi(a_{ij}) = f_{ij} \in C(X)$. Then

$$\|[f_{ij}]\| = \sup_x \sup_{\|\zeta\|, \|\eta\| \leq 1} |\langle [f_{ij}(x)]\zeta, \eta \rangle|,$$

where $\zeta, \eta \in \mathbb{C}^n$. However,

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