ON THE OVALS OF EVEN-DEGREE PLANE CURVES

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1. INTRODUCTION

Let $P(x_0, x_1, x_2)$ be a real homogeneous polynomial of degree d. In 1876, Axel Harnack showed that the real locus $A \subset \mathbb{R}P^2$ of the equation $P = 0$ has at most $g + 1$ components, where $g = \text{genus} = (d - 1)(d - 2)/2$. In the same paper [3], he gave a construction in every degree for a curve with the maximal number of components (an M-*curve*).

If P is nonsingular on $\mathbb{R}P^2$, then A consists of disjoint circles. In even degrees, the *sign* of P is well defined on $\mathbb{R}P^2$, so that each of these circles is two-sided. In odd degrees, exactly one of these circles fails to be two-sided.

Two-sided circles are called *ovals*, and each oval divides $\mathbb{R}P^2$ into a disc and a Möbius band. Thus we can say that one oval lies in another if the first lies in the disc component of the complement of the second.

Our aim here is to publicize some recent Russian work describing the mutual disposition of the ovals of a plane curve, and also to provide an independent proof of an inequality due to V. A. Rohlin.

Let P (respectively, N) be the number of ovals properly contained in an even (respectively, odd) number of ovals. In 1938, I. Petrovsky [4] showed that for a nonsingular curve of even degree $d = 2k$, $|2(P - N) - 1| \leq 3k^2 - 3k + 1$. In particular, this implies that $N \geq (k - 1)(k - 2)/4$.

V. I. Arnol'd [1] has suggested that for M-curves $|P - N - 1| \leq k^2 - 1$, which implies that $N \geq (k - 1)(k - 2)/2$. We give here an inequality, due to Rohlin [8], that is independent of Petrovsky's theorem, but would follow from Arnol'd's inequality.

Let the *complexity* of a curve be the number of proper inclusions among its ovals.

THEOREM. *A nonsingular* M-*curve of even degree* 2k *has complexity at least* $(k - 1)(k - 2)/2$, *and these numbers are equal* mod 2.

In the series of M-curves constructed by Harnack, in odd degrees no oval contains any other oval, while in even degree 2k, there are $g + 1 - (k - 1)(k - 2)/2$ ovals all exterior to each other, and one of these contains the remaining $(k - 1)(k - 2)/2$ ovals, which themselves are mutually exterior to each other. Thus not only is the bound on complexity exact in every even degree, but also there can be no direct analogue of the theorem in odd degrees.

Recently, Arnol'd [1] and Rohlin ([5], [6], and [7]) have proved Gudkov's conjecture [2] that a nonsingular M-curve of even degree 2k satisfies the condition $P - N \equiv k^2 \pmod 8$. Rohlin [8] gives a new proof of the Gudkov conjecture, but only mod 4. The novelty in our proof lies in the fact that it invokes only the geometry of

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