COMPACTNESS CONDITIONS IN TOPOLOGICAL GROUPS

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INTRODUCTION

In 1938, H. Fitting proved the following theorem (see [1] or [3, Section 61]).

THEOREM (H. Fitting). *Every finite group is an extension of a soluble group by a semisimple group.*

Thus the study of the theory of finite groups is reduced to the study of soluble groups, semisimple groups, and the theory of group extensions. Semisimple groups were first defined in the context of finite groups, by Fitting. Later, the idea was extended to infinite groups by P. A. Gol'berg [2]. In the paper [6], the idea was further extended to topological groups.

In this paper, we attempt to find a theorem similar to Fitting's for topological groups. For this purpose, we introduce in Section 1 the concept of W-subgroups of a topological group. In fact, each W-subgroup of a topological group G is a member of a well-ordered ascending soluble chain in G, and it is therefore a generalized soluble group. We shall show that every topological group has a unique maximal W-subgroup, called the W-*radical* of the group. A topological group is called W-*soluble* if it coincides with its W-radical. A structure theorem for W-soluble groups will also be given in Section 1.

Now, for an arbitrary topological group G with W-radical Q, one might expect G/Q to be topologically semisimple (see [6] for the definition of a topologically semisimple group). However, this is not true unless G/Q satisfies a certain minimal condition. This is justified by the following characterization of a topologically semisimple group G:

(a) G has no nontrivial, normal, abelian subgroups, and

(b) every nontrivial, closed, normal subgroup of G has a minimal closed, normal subgroup.

The minimal condition in (b) is certainly satisfied by finite groups. Thus finite semisimple groups can be defined by condition (a) alone. We find that (b) is also satisfied by discrete groups with certain finiteness conditions, for instance, periodic FC-groups (locally normal groups). Thus, if G is a periodic FC-group and Q is its W-radical, then G/Q is semisimple.

This last result leads us to the study of topological groups with an analogous compactness condition. An element in a topological group G is called *bounded* if its class of conjugates is relatively compact. The set of all bounded elements of G, denoted by B(G), forms a normal subgroup of G. An element in G is called *periodic* if it generates a relatively compact subgroup of G. The set of all periodic elements of G is denoted by P(G). In Section 3, we shall describe the structure of a totally disconnected, locally compact group with the compactness condition, $G = P(G) = \overline{B(G)}$, through the study of its W-radical Q and the factor group G/Q. This can also be

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