ON THE SIGNATURE OF FERMAT SURFACES

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1. In this note, we study the signature of certain hypersurfaces of complex projective space. Specifically, we are interested in questions of divisibility of the signature of these hypersurfaces by certain primes; the questions are interesting both because of applications to cobordism theory [9] and because they are related to number-theoretic problems. That such considerations lead to difficult problems in elementary number theory is not surprising. However, it is perhaps unexpected that in certain cases these problems are related to the Fermat conjecture.

Let $\mathbb{C}P^n$ denote complex projective space of dimension n. We define the Fermat surface $Q_n(q)$ to be the hypersurface of complex dimension n and degree q in $\mathbb{C}P^{n+1}$ given by the formula

$$Q_n(q) = \{[z_0, z_1, \cdots, z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_0^q + z_1^q + \cdots + z_{n+1}^q = 0\}.$$

These hypersurfaces admit, in a natural way, an action of an $(n+2)$-dimensional torus, and hence, actions of subgroups of that torus. Study of the equivariant signature under a variety of these actions has led to a number of interesting connections between number theory and topology ([5], [6], [12]). It is our thesis that on an even more elementary level, the study of divisibility of the signature of $Q_n(q)$ by powers of q, where q is an odd prime, leads to interesting questions. In this respect, the present paper is somewhat preliminary; it does not answer the questions, but rather explores the relations between number theory and topology in our special context.

That such relations exist seems to be a consequence of the fact that the signature of the Fermat surfaces can be computed (at least in principle) in divers ways. We point out at the start that although for $q = 2$ and 3 simple formulas for the signature can be obtained, for even modest values of q the numbers involved are quite large, and no simple closed formula, suitable for answering these questions of divisibility, seems to exist. (The signature of $Q_8(11)$ is 48 162 411, while the rank of the middle-dimensional cohomology is 909 090 911.) We have organized this paper from the point of view of obtaining three different formulas for the signature, each with its own purpose.

In Section 2, we briefly sketch the proofs of some well-known facts about the cohomology and Euler characteristic of $Q_n(q)$. Applying the Riemann-Roch theorem, we compute the χ^k-invariants of $Q_n(q)$. Using the Hodge theorem, we therefore establish our first formula for the signature of $Q_n(q)$ as a sum of products of pairs of binomial coefficients. The purpose of this calculation is not so much to investigate divisibility of the signature by powers of q, but to prove a conjecture about $\chi^k(Q_n(q))$. It is easy to compute the Todd genus of $Q_n(q)$ and to see that when $n \geq q - 1$ it is 1. This implies that $h^{0,n}(Q_n(q)) = 0$ when $n \geq q - 1$ (see [4] for notation.) Hence one might conjecture that for fixed k, $\chi^k(Q_n(q)) = (-1)^k$ and

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