HADAMARD'S INEQUALITY FOR MATRICES WITH POSITIVE-DEFINITE HERMITIAN COMPONENT

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Hadamard's inequality states that if $A = (a_{ij})$ is an n-by-n positive-definite Hermitian matrix, then

$$\det A \leq a_{11} a_{22} \cdots a_{nn}.$$

If we let $\Pi_n \equiv \{A \in M_n(C): A + A^* > 0\}$, then $A \in \Pi_n$ does not necessarily satisfy the analogous inequality

(1) $$|\det A| \leq |a_{11} \cdots a_{nn}|.$$

D. M. Koteljanskiĭ [3], F. R. Gantmacher and M. G. Krein [2], and K. Fan [1] have generalized the Hadamard inequality by isolating a class (which includes the positive-definite Hermitian matrices) throughout which (1) holds. In this note, we point out a different class (related to a convexly parametrized subclass of Π_n) in which (1) holds, and, in the process, also give a dual class in which

(2) $$|\det A| \geq |a_{11} \cdots a_{nn}|$$

holds. The former class also includes the positive-definite Hermitian matrices, so that an alternate proof of Hadamard's inequality is provided.

For $A \in M_n(C)$, we define $H(A) = \frac{A + A^*}{2}$ and $S(A) = \frac{A - A^*}{2}$, so that $A = H(A) + S(A)$. If $H = (h_{ij})$ is Hermitian, we define the *upper triangular part* of H by $T(H) \equiv (t_{ij})$ where

$$t_{ij} = \begin{cases} 2h_{ij} & \text{if } i < j, \\ h_{ij} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

For $A \in M_n(C)$, we then define $T(A)$ by $T(A) \equiv T(H(A))$. It also follows from our definitions that $H(T(A)) = H(A)$. In this context, Hadamard's inequality states simply that

$$\det H \leq \det T(H)$$

when H is positive-definite Hermitian.

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