A NORM INEQUALITY IN HYPONORMAL OPERATOR THEORY

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1. INTRODUCTION

Recall that a bounded operator T on a Hilbert space $\mathfrak{H}$ is *hyponormal* if

$$T^*T - TT^* = D \geq 0, \tag{1.1}$$

and *completely hyponormal* if, in addition, there is no nontrivial subspace on which T is normal. If $T = H + iJ$ is the Cartesian representation of T, then (1.1) is equivalent to

$$HJ - JH = -iC, \quad \text{where } D = 2C \geq 0. \tag{1.2}$$

It is known that the spectra of H and J are the (real) sets obtained by projecting the spectrum $\sigma(T)$ of T onto the x- and y-axes; see [2, p. 46]. Also, by [3],

$$2\pi \|C\| \leq \text{meas}_2(\sigma(T)). \tag{1.3}$$

Further, if $H = \Re(T)$ has the spectral resolution

$$H = \int t\, dE_t, \tag{1.4}$$

and if T is completely hyponormal, then the spectral family $\{E(\cdot)\}$ is strongly absolutely continuous, that is, $\|E_t f\|^2$ is absolutely continuous in t for each f in $\mathfrak{H}$; see [2, pp. 20, 42].

If α is a Borel set on the real line, then $T_\alpha = E(\alpha)\, T\, E(\alpha)$ is hyponormal, in fact, $T_\alpha^* T_\alpha - T_\alpha T_\alpha^* = E(\alpha)\, D\, E(\alpha) \geq 0$. If $\alpha = \Delta$ is an open interval, and if $E(\alpha) \neq 0$, it follows from the results of [4] that

$$\sigma(T_\Delta) = (\sigma(T) \cap \{z\colon \Re(z) \in \Delta\})^-, \tag{1.5}$$

where $T_\Delta = E(\Delta)\, T\, E(\Delta)$ is regarded as an operator on $E(\Delta)\mathfrak{H}$. Since $\sigma(E(\Delta)\, J\, E(\Delta))$ is the projection of $\sigma(T_\Delta)$ onto the y-axis, one easily obtains from (1.5) the norm of $E(\Delta)\, J\, E(\Delta)$ (as an operator either on $\mathfrak{H}$ or on $E(\Delta)\mathfrak{H}$) in terms of the spectrum of T in the form

$$\|E(\Delta)\, J\, E(\Delta)\| = \sup\{|\Im(z)|\colon z \in \sigma(T) \text{ and } \Re(z) \in \Delta\}. \tag{1.6}$$

If F(t) denotes the linear measure of the intersection of $\sigma(T)$ with the line $\Re(z) = t$, so that

$$F(t) = \text{meas}_1[\sigma(T) \cap \{z\colon \Re(z) = t\}], \tag{1.7}$$

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