SOME REMARKS ON ERGODICITY AND INVARIANT MEASURES

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Dedicated to Prof. H. Hornich on the occasion of his 70th birthday.

1. INTRODUCTION

The main goal of this paper is to describe a method, applicable to a large class of noninvertible transformations, for proving ergodicity and existence of an invariant measure. We introduce an auxiliary transformation that behaves neatly under some conditions met in applications. This device first appeared in papers by R. Fischer [2] and Schweiger [10], but the approach given in this paper is much more general. For example, we include transformations of the type considered by H. Jager [4] for the very special case of decimal expansions. The model employed is as follows: A pair (M, S), where M is a set and $S: M \to M$ is a map, is called a *fibered system* if there exist a finite or countable set I and a partition $\{B(i) \mid i \in I\}$ of M such that the restriction of S to any B(i) is injective. Let N denote the set of natural numbers, and let X be the set of all functions $f: N \to I$; then the transformation $T: X \to X$, $(Tf)(n) = f(n+1)$ is called the *shift*. The map $\Phi: M \to X$ defined by $\Phi(m)(n) = i$ if $S^{n-1} m \in B(i)$ gives rise to the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & X \\ S \downarrow & & \downarrow T \\ M & \xrightarrow[\Phi]{} & X \end{array}$$

If Φ is injective (that is, one-to-one), we call Φ *valid* (after W. Parry [5]).

In Section 2, we give the construction of the auxiliary fibered system (M^*, S^*). We show that the validity of Φ implies the validity of the corresponding Φ^*. Section 3 contains a proof that if S^* is ergodic, so is S.

In Section 4, we show that for each invariant measure for S^* one can write down an explicit formula for an invariant set function for S that turns out to be a finite or σ-finite measure in a great number of cases. In Section 5, we discuss the connection between this construction and induced transformations. In particular, we compare the different approaches to Boole's transformation (R. L. Adler and B. Weiss [1], Schweiger [10]).

2. THE CONSTRUCTION

A *cylinder* of rank n is a set

$$B(i_1, \cdots, i_n) = B(i_1) \cap S^{-1} B(i_2) \cap \cdots \cap S^{-n+1} B(i_n).$$

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