WEAK COMPLETENESS AND INVARIANT SUBSPACES

Donald W. Hadwin

Throughout, X will denote a complex locally convex topological vector space. An *operator* on X is a continuous linear transformation on X. An operator on X is *intransitive* if it has a non-trivial closed invariant subspace. The space X is *operator-intransitive* if every operator on X is intransitive. A *hyperinvariant subspace* of an operator T is a subspace that is left invariant by every operator commuting with T.

A. L. Shields [5, Theorem 2] showed that the space (s) of all complex sequences (topologized by the coordinate seminorms) is operator-intransitive. Later, B. E. Johnson and A. L. Shields [1, Theorem 1] proved that every operator on (s) that is not a scalar has a non-trivial closed hyperinvariant subspace.

Since (s) is a separable, locally convex Fréchet space, it seems natural to ask whether the techniques applied to (s) might also be applied to some infinite-dimensional Banach space.

This paper isolates the property (property #) that makes Shields's proof [5, Theorem 2] work, and it shows (Theorem 1) that this property is equivalent to weak completeness. (Note: weak completeness means that every weakly fundamental net is convergent.)

The remainder of this paper shows (Theorem 2), for a weakly complete space X with dimension greater than 1, that every operator on X that is not a scalar has a non-trivial hyperinvariant subspace if and only if the (continuous) dual of X has linear dimension less than $2^{\aleph_0}$.

The notation and terminology of [3] will be used. The set of all linear functionals on X will be denoted by X^*, and the set of those functionals in X^* that are continuous will be denoted by X'. Also, dim X will denote the linear dimension of X. If M is a subspace of X', then $M^{\perp}$ denotes the set of all vectors in X that annihilate M. The space X has *property* # provided that, for every subspace M of X', $M^{\perp} = 0$ only if $M = X'$.

The proof of the following proposition is almost a word-for-word copy of Shields's proof that (s) is operator-intransitive.

PROPOSITION. *If* X *has property # and* $\dim X > 1$, *then* X *is operator-intransitive.*

Proof. Let T be an operator on X. Then T' (the adjoint of T) is a linear transformation on X'. By a theorem of H. H. Schaefer [4], there is a subspace M of X' such that $0 \neq M \neq X'$ and $T'(M) \subseteq M$. Therefore $M^{\perp}$ is a closed subspace of X and $T(M^{\perp}) \subseteq M^{\perp}$. Since $M \neq 0$, we see that $M^{\perp} \neq X$. Since X has property # and $M \neq X'$, it follows that $M^{\perp} \neq 0$. Thus T is intransitive.

Let $\mathbb{C}$ denote the field of complex numbers. If B is a nonempty set, let

$$\mathbb{C}^B = \{\phi \colon \phi \text{ is a function from } B \text{ to } \mathbb{C}\}$$

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