STEENROD SQUARES AND REDUCTION OF STRUCTURE GROUP FOR FIBRATIONS

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INTRODUCTION

All spaces considered in this paper will be assumed to have the homotopy types of regular CW-complexes with integral homology of finite type. Z_2 coefficients will be used for homology and cohomology, and A will denote the mod-2 Steenrod algebra. We shall regard two principal fibrations $G \to E \to B$ and $G' \to E' \to B'$ as fiber-homotopically equivalent as principal fibrations, if there is a homotopy equivalence $G \to G'$ that is a homomorphism and is compatible with a fiber-homotopy equivalence of $E \to B$ with $E' \to B'$. We shall be concerned with properties of principal fibrations that are invariant under modifications of the type just described. The diagram $G \to E \xrightarrow{\pi} B$ will therefore be used to denote any suitably chosen representative of such a fiber-homotopy equivalence class of principal fibrations with simply connected bases. For a representative that is a fiber bundle, the structure group will be regarded as acting on the right on E. For a subgroup H of G, $H \backslash G$ will denote the space of right cosets of H, and G/H the space of left cosets.

$E(\pi)$ will denote the Z_2 Serre cohomology spectral sequence of the fibration π, with $E_2^{p\,q} = H^p(B) \otimes H^q(G)$. In [1], [8, Section 4], [9], and [13], natural operations

$$S^i: E_r^{p\,q} \to E_{2r-2}^{p+i-q\;2q} \qquad (i \geq q,\; 2 \leq r \leq \infty)$$

have been defined. These are compatible with the differentials, and they are compatible on E_∞ with the action of the Steenrod algebra A on $H^*(E)$. If $b \otimes g \in E_2^{p\,q}$, then $S^i(b \otimes g) = Sq^{i-q} b \otimes g^2$. Of course, for any fibration, the action of A on the cohomology of the total space, the base space, and the fiber must be compatible with the homomorphisms induced by the projection and the inclusion of the fiber in the total space. Because of the properties of the S^i just mentioned, the S^i clearly impose additional restrictions upon the choices of the three spaces for a fibration.

In this paper, we consider two cases in which, for the fiber bundle $G \to E \xrightarrow{\pi} B$, the assumption that the structure group G is reducible to a subgroup H leads to a factorization of π of the form $E \xrightarrow{\bar{\pi}} B \times (H \backslash G) \to B$, where $\bar{\pi}$ is also a fiber bundle. In the first case, H is normal; in the second case, $G \to H \backslash G$ is itself a fiber bundle. The above-mentioned properties of the S^i can therefore be employed with some choices of π to show that such a reduction of the structure group is impossible.

For principal bundles over spheres, both types of reduction of the structure group are worthy of consideration. As we shall see, the high connectivity of the sphere leads to a reduction to an advanced stage in the Postnikov resolution of the

Received March 9, 1973.
The author thanks Princeton University for financial support at the time he undertook this research.

Michigan Math. J. 22 (1975).