DISTRIBUTIVE NOETHER LATTICES

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1. INTRODUCTION

R. P. Dilworth [4] introduced the concept of a Noether lattice as an abstraction of the concept of the lattice of ideals of a Noetherian ring. In [2], K. P. Bogart showed that a distributive regular local Noether lattice of dimension n is isomorphic to the sublattice RL_n of ideals of $F[X_1, \cdots, X_n]$, where F is a field, generated by the principal ideals $(X_1), \cdots, (X_n)$ under multiplication and joins. In [3], Bogart further showed that each distributive local Noether lattice is a certain quotient of RL_n. E. W. Johnson and J. P. Lediaev [7] characterized the distributive Noether lattices that can be represented as the lattice of ideals of a Noetherian ring.

By a semigroup, we shall mean a commutative semigroup with 0 and 1, written multiplicatively. In Section 2 we show that the lattice of ideals of a semigroup is a quasi-local distributive multiplicative lattice, and we give a necessary and sufficient condition for a multiplicative lattice to be the lattice of ideals of a semigroup. As a corollary, it follows that a distributive local Noether lattice is the lattice of ideals of a certain type of semigroup. In Section 3 we show that every distributive local Noether lattice can be embedded in the lattice of ideals of a Noetherian ring. In Section 4 we consider distributive regular Noether lattices. Two characterizations of distributive regular Noether lattices are given. An interesting result proved in this section is that if in a principally generated distributive multiplicative lattice two primes are not comaximal, then their join is again prime.

2. SEMIGROUPS

Let L be a multiplicative lattice. An element M of L is *meet* (*join-*) *principal* if $AM \wedge B = (A \wedge (B:M))M$ (if $(AM \vee B:M) = A \vee (B:M)$) for all A and B in L. An element M of L is *weak-meet* (*weak-join*) *principal* if

$$M \wedge B = (B:M)M \qquad ((BM:M) = B \vee (0:M))$$

for all B in L. A *principal element* is an element that is both meet-principal and join-principal. We say that a multiplicative lattice is *quasi-local* if it has a unique maximal element.

Let S be a commutative semigroup with 0 and 1, written multiplicatively. The set-theoretic union or intersection of each set of ideals is again an ideal. Thus L(S), the lattice of ideals of S, is easily seen to be an infinitely distributive multiplicative lattice. (Recall that a lattice L is said to be infinitely distributive if $A \wedge \left(\bigvee_\alpha B_\alpha\right) = \bigvee_\alpha (A \wedge B_\alpha)$ for all $A \in L$ and $\{B_\alpha\} \subseteq L$, where $\{\alpha\}$ is an arbitrary indexing set.) The set of nonunits of S forms the unique maximal ideal of S, so that L(S) is quasi-local. Each principal ideal (x) of S is meet-principal in L(S); however, it need not be weak-join-principal. For example, if $S = \{0, 1, x, y\}$

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