NILPOTENT ELEMENTS OF COMMUTATIVE SEMIGROUP RINGS

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INTRODUCTION

All rings considered in this paper are assumed to be commutative, all semigroups are assumed to be abelian, the semigroup operation is written as addition, and the existence of a zero element with respect to this operation is assumed. On the other hand, the assumption that the rings under consideration have an identity plays no essential role, and therefore it will not be made.

We are concerned with the problem of determining the set of nilpotent elements of the semigroup ring of a semigroup S over a ring R. We follow the notation of D. G. Northcott [9, p. 128] and write R[X; S] for the semigroup ring of S over R; the elements of R[X; S] are "polynomials" $r_1 X^{s_1} + \cdots + r_n X^{s_n}$ in X with coefficients in R and exponents in S. If N is the nilradical of R, then it is clear that N[X; S] is contained in the nilradical of R[X; S]; this containment may be proper, and it depends upon the presence of certain torsion in the semigroup S.

After disposing of certain preliminaries concerning semigroups and semigroup rings in Section 1, we determine in Section 2 the nilradical of R[X; S] for the case where R is a ring of nonzero characteristic n. Let p be a prime integer; elements s and t of S are said to be *p-equivalent* if $p^k s = p^k t$ for some positive integer k, and S is *p-torsion-free* if distinct elements of S are not p-equivalent. In Theorem 2.5 we prove that the set of nilpotent elements of R[X; S] is N[X; S] + I, where I is the ideal generated by the set

$$\{rX^a - rX^b \mid r \in R, \text{for some prime divisor } p_i \text{ of n the element a is } p_i\text{-equivalent to b, and a power of } p_i \text{ annihilates } r\};$$

thus N[X; S] is the nilradical of R[X; S] if and only if S is p_i-torsion-free for each prime divisor p_i of n.

In Section 3 we take up the case of a ring of characteristic 0. The existence of nilpotent elements of R[X; S] - N[X; S] is closely related to, but not equivalent to, the presence of asymptotically equivalent elements of S, where the definition is as follows. Elements a and b of S are *asymptotically equivalent* if there exists a positive integer n_0 such that na = nb for each $n \geq n_0$. If R is an integral domain, then R[X; S] has nonzero nilradical if and only if S contains nonidentical asymptotically equivalent elements. (Theorems 3.6 and 3.9.) In general, the nilradical of

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