THE STRUCTURE OF JORDAN H-ALGEBRAS

J. A. Loustau

1. INTRODUCTION

In [3], I. N. Herstein considered an associative ring R with the property that for each x in R there exists a polynomial $p_x(X)$ with integral coefficients, in an indeterminant X, such that $x^2 p_x(x) - x$ lies in the center of R, and he proved that every associative ring with this property is commutative. Further, Herstein and S. Montgomery [6] have extended Herstein's result by proving that if an associative division ring with involution has the property that to each symmetric element $x = x^*$ there corresponds an integer $n > 1$ such that $x^n - x$ is central, then the ring is commutative or four-dimensional over its center. In [1], W. Burgess and M. Chacron have demonstrated that in an associative ring R with involution, the property that $x^2 p_x(x^2) - x$ is central for each symmetric element implies that R is an integral extension of its center of degree not larger than 2. It is our aim to determine the structure of a Jordan algebra over a field of characteristic not 2 that satisfies the condition of Herstein's theorem [3]. In particular, we shall call a Jordan algebra J a *Jordan* H-*algebra* if for every x in J there exists a polynomial $p_x(X)$, in an indeterminant X with integral coefficients such that $x^2 p_x(x) - x$ lies in the center of J. The following result is our main theorem.

THEOREM 1. *Let* J *be a Jordan* H-*algebra over a field of characteristic not two; then* J *is isomorphic to a subdirect sum of Jordan algebras* $\{J_i\}_{i \in \Lambda}$, *where each* J_i *is either an associative algebra or a simple periodic Jordan algebra of capacity two.*

We remark that in [7], an associative ring R is called an H-ring if for every x in R there exists an integer $n(x) > 1$ such that $x^{n(x)} - x$ is in the center of R. However, from results in [3] and [2, p. 220] it follows immediately that this condition and the centrality of $x^2 p_x(x) - x$ are equivalent. Though we have not proved this equivalence for Jordan algebras, we shall use here the more general condition to denote a Jordan H-algebra.

As corollaries to Theorem 1 we shall prove that if J is as in Theorem 1, then J is associative if either its idempotents are central, or it contains no idempotents different from zero or one, or it contains a unique nonzero idempotent. Further, we shall prove analogues, for Jordan algebras over a field of characteristic not 2, of results given in [4] and [9].

The proof of Theorem 1 will proceed as follows. After some preliminary lemmas, we shall reduce the problem to one of determining the structure of a subdirectly irreducible Jordan algebra satisfying the hypothesis of Theorem 1. Next, the subdirectly irreducible case will be divided into two cases, depending on whether the algebra contains an idempotent different from zero and the identity, that is, whether the algebra contains a nontrivial idempotent. When the subdirectly irreducible algebra contains nontrivial idempotents, the algebra will be a simple periodic Jordan algebra of capacity 2. The proof of this will proceed similarly to the proof

Received October 26, 1973.

Michigan Math. J. 22 (1975).