GEODESIC SPHERES AND SYMMETRIES IN NATURALLY REDUCTIVE SPACES

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INTRODUCTION

In [2], the author and H. K. Nickerson proved that in a naturally reductive pseudo-Riemannian homogeneous space, the geodesic symmetries are divergence-preserving (volume-preserving up to sign). The proof is based on a complicated combinatorial identity; the paper [2] also contains a proof, due to N. Wallach, which is much simpler but not obviously applicable to all naturally reductive spaces. In Section 1 of the present paper, we prove an algebraic result, and in Section 2, we use it to extend Wallach's proof to all naturally reductive spaces. We find other restrictions on the geometry of naturally reductive spaces (respectively, harmonic spaces); in particular, in Section 3, assuming a positive-definite metric, we show that all sufficiently small geodesic spheres have antipodally symmetric (respectively, constant) mean and scalar curvatures. There is overlap here with work of S. Tachibana and T. Kashiwada [6].

1. ALGEBRAIC PRELIMINARIES

Let $\mathfrak{g}$ be a Lie algebra over a field F of characteristic different from 2, and suppose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a vector-space decomposition, where $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Let P and K denote projection on $\mathfrak{p}$ and $\mathfrak{k}$, respectively, and let B be a symmetric, bilinear F-valued form on $\mathfrak{p}$ satisfying the condition

$$(1) \qquad B(P[V, Y], Z) + B(Y, P[V, Z]) = 0 \qquad \text{for } V \in \mathfrak{g},\ Y, Z \in \mathfrak{p}.$$

LEMMA. *For* $X \in \mathfrak{p}$, *the map* $\operatorname{ad} X \circ K \circ \operatorname{ad} X\colon \mathfrak{p} \to \mathfrak{p}$ *is* B-*symmetric.*

Proof. For $X, Y, Z \in \mathfrak{p}$, we have the relations

$$\begin{aligned} B([X, K[X, Y]], Z) &= -B(X, [Z, K[X, Y]]) \\ &= -B(X, P[Z, [X, Y]]) + B(X, P[Z, P[X, Y]]), \\ -B(X, P[Z, [X, Y]]) &= B(X, P[X, [Y, Z]]) + B(X, P[Y, [Z, X]]) \\ &= B(X, P[Y, P[Z, X]]) + B(X, P[Y, K[Z, X]]) \\ &= -B(P[X, Y], P[X, Z]) + B([X, K[X, Z]], Y). \end{aligned}$$

Remark 1. In the situation of Section 2, the lemma is equivalent to the curvature identity $\langle R(X, Y)X, Z \rangle = \langle Y, R(X, Z)X \rangle$.

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