CENTRALIZERS OF SEPARABLE SUBALGEBRAS

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0. INTRODUCTION

There has been some interest recently in the relationship between the structure of an algebra R and the centralizer of an appropriate subset A of R (denoted by $C_R(A)$). In particular, I. N. Herstein and L. Neuman have considered the case when A consists of a single element $a \in R$ such that a^n is in the center of R for some positive integer n. They have shown that if R is semiprime and $C_R(a)$ is simple or semisimple Artinian, then R itself must also be simple or semisimple Artinian. This result was extended by M. Cohen [2], who showed that if $C_R(a)$ is a Goldie ring, then R must also be a Goldie ring.

The intent of this paper is to show that these results can be extended to the situation where A is any finite-dimensional separable subalgebra of R. That is, we show that if R is semiprime and $C_R(A)$ is either simple, semisimple, or semisimple Artinian, then so is R.

Further results are obtained on the relationships between the ideals, zero-divisors, and Jacobson radical of $C_R(A)$ and those of R. The result on zero-divisors has now been used by Cohen to show that if R is semiprime and $C_R(A)$ is a Goldie ring, then so is R [3]. We discuss these results in more detail at the end of the paper.

We note that centralizers of separable subalgebras arise naturally as fixed-point sets of automorphism groups, as follows: Let R be a ring whose center k is a field, and let G be a finite group of inner automorphisms of R as a k-algebra such that the order of G is relatively prime to the characteristic of k. For each $\tau \in G$, choose an $x_\tau \in R$ that induces τ. If A is the subalgebra of R generated by the x_τ, then $C_R(A)$ is precisely the ring of fixed points R^G of G acting on R. The algebra A is separable, since it is a homomorphic image of a twisted group algebra $k_t[G]$, which is separable. (For details, see [7].)

This relationship was used in [7] to show that if R^G satisfies a polynomial identity (PI), then R also satisfies an identity, where G is as described above. For, it was first shown that if A is a finite-dimensional separable subalgebra of a k-algebra R, and $C_R(A)$ satisfies a PI, then R satisfies a PI.

1. PRELIMINARIES

In all that follows, unless otherwise stated, R will denote an algebra over a field k, and A will denote a finite-dimensional, separable k-subalgebra of R. By J(R) we denote the Jacobson radical of R, and by N(R) the lower nil (prime) radical of R. A ring R is said to be semiprime of N(R) = (0); equivalently, R has no nonzero nilpotent ideals. $C_R(A)$ denotes the centralizer of A in R.

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