# A MODEL FOR QUASINILPOTENT OPERATORS

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Whether every quasinilpotent operator on a separable, infinite-dimensional, complex Hilbert space has a nontrivial invariant (hyperinvariant) subspace is and has long been a stubborn and intractable open question. The purpose of this note is to establish the existence of a model (up to similarity) for such operators (Theorem 1), and to discuss some of the consequences of the existence of this model. It is believed that these results are pertinent to the invariant-subspace problem mentioned above.

We begin by recalling some notation and terminology. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be separable, infinite-dimensional, complex Hilbert spaces. If X: $\mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear transformation such that kernel X = kernel X* = $\{0\}$, then X is called a *quasiaffinity*. If $A_1$ and $A_2$ are bounded operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and there exists a quasiaffinity X: $\mathscr{H}_1 \to \mathscr{H}_2$ such that $XA_1 = A_2 X$, we say that $A_1$ is a *quasiaffine transform* of $A_2$, and we write $A_1 \prec A_2$. If $A_1$ and $A_2$ are quasiaffine transforms of each other, that is, if there exist quasiaffinities X: $\mathscr{H}_1 \to \mathscr{H}_2$ and Y: $\mathscr{H}_2 \to \mathscr{H}_1$ such that $XA_1 = A_2 X$ and $A_1 Y = YA_2$, then $A_1$ and $A_2$ are said to be *quasisimilar*. It is known that if $A_1$ and $A_2$ are quasisimilar operators, and $A_1$ has a nontrivial hyperinvariant subspace, then so does $A_2$ (see [2], [4], [7]). In the remainder of the paper, $\mathscr{H}$ will always be a separable, infinite-dimensional, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$. Recall that an operator A is a *part* of an operator B if A is the restriction of B to some invariant subspace of B. The following structure theorem seems interesting and has some noteworthy consequences.

THEOREM 1. *Let* T *be a quasinilpotent operator in* $\mathscr{L}(\mathscr{H})$. *Then there exists a compact* (*quasinilpotent*) *weighted backward shift* K *in* $\mathscr{L}(\mathscr{H})$ *such that* T *is similar to a part of the operator* $K \oplus K \oplus \cdots \oplus K \oplus \cdots$ *acting on the direct sum of countably many copies of* $\mathscr{H}$.

*Proof.* We treat only the case that T is not nilpotent. The case in which T is nilpotent follows by an obvious modification of our argument below. Consider the (well-defined) sequences

$$\alpha_n = \|T^n\|^{1/2} \quad (n = 0, 1, 2, \cdots), \tag{1}$$

$$\omega_n = \alpha_n/\alpha_{n-1} \quad (n = 1, 2, 3, \cdots), \tag{2}$$

and observe that they satisfy the conditions

$$0 < \alpha_{m+n} \leq \alpha_m \alpha_n, \tag{3}$$

$$\alpha_n^{1/n} \to 0, \tag{4}$$

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