ON APPROXIMATION OF NORMAL OPERATORS BY WEIGHTED SHIFTS

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An intriguing problem in the theory of perturbations of operators on a separable Hilbert space is the following: If an operator has a small self-commutator and there are no points in the spectrum of the operator of nonzero Fredholm index, is that operator necessarily close in norm to a normal operator? That is, is there a norm analogue to the result of L. G. Brown, R. G. Douglas, and P. A. Fillmore on operators with compact self-commutators? If the self-commutator is both small and compact, must the operator differ from a normal operator by a small compact operator?

In this note we show that if a weighted bilateral shift on a separable Hilbert space has a small self-commutator and has no point in its spectrum on which the shift has a Fredholm index other than zero, then that shift is close in norm to a normal operator. If, in addition, the self-commutator is compact, then the difference between the shift and the normal operator is compact and of small norm.

To be precise, we show the following (see Theorem 2).

Let $\{\phi_i\}$ $(-\infty < i < \infty)$ be an orthonormal basis for H. Let S be the weighted bilateral right shift defined by $S(\phi_i) = a_i \phi_{i+1}$. Let $\|S\| = 1$. Now assume that $\{a_i\}$ satisfies the following two conditions:

Condition 1 (no nonzero index).

$$\liminf_{i \to -\infty} |a_i| \leq \limsup_{i \to \infty} |a_i|, \qquad \liminf_{i \to \infty} |a_i| \leq \limsup_{i \to -\infty} |a_i|.$$

Condition 2 (small self-commutator).

For some $\varepsilon < 1/256$, the inequality

$$\| a_i | - | a_{i+1} \| < \varepsilon,$$

holds for each index i.

Then there exists a normal operator N such that $\|N - S\| < 100\sqrt{\varepsilon}$.

The estimate in the conclusion is not sharp. We are interested primarily in the fact that if Condition 1 holds, then a shift with a small self-commutator is close to a normal operator. In this form, Theorem 2 is best possible, except for the lack of sharpness in the estimate.

Condition 1 is clearly necessary. For if, say,

$$\liminf_{i \to -\infty} |a_i| = \limsup_{i \to \infty} |a_i| + \eta \qquad \text{for some } \eta > 0,$$

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