EXTREMAL PROBLEMS IN ARBITRARY DOMAINS, II

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1. INTRODUCTION

This paper deals with the extremal functions of a certain class of extremal problems. We obtain uniqueness of the extremal functions associated with a class of extremal problems, including the problems treated by D. Hejhal in [7]. We also study the behavior of the extremal function near a free analytic boundary arc. Our techniques are adapted from [4]; they are the techniques of function algebras, and they offer a perspective that is "dual" in some sense to the classical approach of S. Ja. Havinson [6] and others.

In order to state explicitly the main results, we fix some notation.

Let D be a bounded domain in the complex plane $\mathbb{C}$, let K be a compact subset of D, and let η be a measure on K. Let u be a continuous real-valued function on D. The basic extremal problem is the following:

(1.1) To maximize $\left|\int f\, d\eta\right|$, among all analytic functions f on D such that $|f| \leq e^u$ on D.

The extremal problem (1.1) is *nontrivial* if there exists a competing function f for which $\int f\, d\eta \neq 0$.

A normal-families argument shows that there exists an extremal function F for (1.1). Upon multiplying F by a unimodular constant, we can arrange that $\int F\, d\eta \geq 0$. Such an extremal function is said to be *normalized*.

An example of Hejhal [7, p. 114] shows that the normalized extremal function for (1.1) need not be unique, even if u is harmonic. One of Hejhal's uniqueness theorems can be stated as follows.

THEOREM 1.1. *Suppose that* u *is harmonic on* D *and that every component of* $D \setminus K$ *includes in its boundary an essential boundary point of* D. *If the extremal problem* (1.1) *is nontrivial, then the problem has a unique normalized extremal function.*

Hejhal's proof of Theorem 1.1 depends on the methods developed by Havinson [6], who proved the uniqueness of the Ahlfors function of arbitrary domains. Now there is in [4] an economical proof of Havinson's theorem that depends on function-algebraic techniques (see also [3] and [5]). In Section 2, we show how this simple proof can be extended to include Theorem 1.1. Sections 3 and 4 include assorted extensions of the basic uniqueness theorem. In particular, a result obtained in Section 3 includes the various uniqueness assertions of [7].

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