EXISTENCE OF SOLUTIONS OF A NONLINEAR PROBLEM IN POTENTIAL THEORY

R. Kannan

1. INTRODUCTION

In this paper we study the nonlinear boundary value problem

$$(1)\qquad \begin{cases} \Delta u + g(x, y, u) = 0, & (x, y) \in A = \{x^2 + y^2 < 1\}, \\ u = 0, & (x, y) \in \partial A = \{x^2 + y^2 = 1\} \end{cases}$$

under various hypotheses on g. We denote by N the Nemitsky operator defined by $Nu = -g(x, y, u(x, y))$, and we assume that $N: S \to S$ maps the space $S = L_2(A)$ into itself. We obtain the following results:

I. If $N: S \to S$ is monotone and continuous, then the nonlinear problem (1) has a unique solution.

II. If $-N: S \to S$ is monotone, continuous, and bounded, and if the Gateaux derivative of N "lies between two consecutive eigenvalues" of the associated linear problem

$$\Delta u + \lambda u = 0 \text{ in } A, \qquad u = 0 \text{ on } \partial A,$$

then the problem (1) has a unique solution.

III. If $Nu = \lambda_m u + h(x, y, u)$, where λ_m is an eigenvalue of the associated linear problem (the resonance case), and the Nemitsky operator $M: S \to S$ defined by $Mu = h(x, y, u)$ is continuous and bounded in S, then under suitable hypotheses the nonlinear problem (1) has at least one solution.

This paper was motivated by a paper of L. Cesari [3] concerning problem (1), where use is made of the alternative method by means of which the problem is reduced to an equivalent system of two operator equations. This method, which has its origin in Lyapunov and Schmidt's work, was formulated by Cesari [2] in functional-analytic terms. The method was then applied by Cesari and several other authors to a wide variety of situations (see J. K. Hale [6]). In this paper we follow this method, but we appeal to several concepts of nonlinear functional analysis, namely maximal monotone operators, nonlinear Hammerstein equations, and Schauder's principle of invariance of domain.

The chief feature of this method is that one can handle problems of the type (1) where the linear operator has a nontrivial nullspace (which is the case if for example $g(x, y, u) = \lambda u + h(x, y, u)$, where λ is an eigenvalue of the associated linear problem $\Delta u + \lambda u = 0$ in A and $u = 0$ on ∂A). As will be obvious from the proofs, the nonlinear problem (1) could be stated in a more general form for elliptic problems in more general domains. However, for the sake of simplicity, we shall restrict ourselves to problem (1).

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