TWO THEOREMS ON KAEHLER MANIFOLDS

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1. INTRODUCTION

Let N be an n-dimensional submanifold (in this paper, we consider only manifolds of real dimension) of a 2m-dimensional Kaehler manifold M with complex structure J and Riemannian metric g, and let $\tilde{\nabla}$ and ∇ be the covariant differentiations on M and N, respectively. Then the second fundamental form σ of the immersion is defined by the equation $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to N, and where σ is a normal-bundle-valued symmetric 2-form on N. For a vector field ξ normal to N, we write

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential component (respectively, the normal component) of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be *parallel* if $D_X \xi = 0$ for each vector field X tangent to N. The submanifold N is said to be *totally umbilical* if $\sigma(X, Y) = g(X, Y)H$, for all vector fields X and Y tangent to N, where $H = (1/n)\,\mathrm{trace}\,\sigma$ is the mean-curvature vector of N in M. In particular, if the second fundamental form σ vanishes identically, N is called a *totally geodesic* submanifold of M. The submanifold N is called a *holomorphic submanifold* (respectively, a *totally real submanifold*) of M if each tangent space of N is mapped into itself (respectively, into the normal space) by the complex structure J.

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex-space-form*, and a Riemannian manifold of constant sectional curvature is called a *real-space-form*.

In his book on Riemannian geometry, É. Cartan [1, p. 231] proved that an n-dimensional, totally umbilical submanifold of a euclidean m-space is either an n-plane or an n-sphere (for more general cases, see [2, p. 50] for example). In Section 3, we shall prove the following result.

THEOREM 1. *Let* N *be an* n*-dimensional, totally umbilical submanifold* $(n \geq 2)$ *of a* 2m*-dimensional complex-space-form* M *of holomorphic sectional curvature* $c \neq 0$. *Then* N *is one of the following submanifolds*:

(a) *a complex-space-form immersed holomorphically in* M *as a totally geodesic submanifold, or*

(b) *a real-space-form immersed in* M *as a totally real and totally geodesic submanifold, or*

(c) *a real-space-form immersed in* M *as a totally real submanifold with nonzero parallel mean-curvature vector.*

Case (b) *occurs only when* $m \geq n$, *and case* (c) *occurs only when* $m > n$.

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