COMPACT FAMILIES OF UNIVALENT FUNCTIONS AND THEIR SUPPORT POINTS

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1. INTRODUCTION

Let D be a plane domain and H(D) the space of analytic functions on D endowed with the topology of locally uniform convergence. It is well known that H(D) is a metrizable, locally convex topological vector space; we denote by H'(D) its topological dual space and by $H_u(D)$ the set of univalent functions in H(D). We shall be interested in subsets of $H_u(D)$ whose elements are normalized by two continuous linear functionals, that is, in subsets of the form

$$(1) \qquad \mathscr{F} = \mathscr{F}(D, \ell_1, \ell_2, P, Q) = \{f \in H_u(D): \ell_1(f) = P, \ \ell_2(f) = Q\},$$

where ℓ_1 and ℓ_2 denote fixed functionals in H'(D) and P and Q denote points in **C**.

A number of the standard families of univalent functions are of the form (1). We single out two:

(i) Let $z_0 \in D$ and $\ell_1(f) = f(z_0)$, $\ell_2(f) = f'(z_0)$, $P = 0$, $Q = 1$. Then

$$(2) \qquad \mathscr{S}(D, z_0) = \{f \in H_u(D): f(z_0) = 0, \ f'(z_0) = 1\}$$

is of the form (1). For $U = \{z: |z| < 1\}$, the set $S = \mathscr{S}(U, 0)$ is the familiar normalized schlicht class.

(ii) Let $p, q \in D$ $(p \neq q)$, $P, Q \in \mathbf{C}$ $(P \neq Q)$, and $\ell_1(f) = f(p)$, $\ell_2(f) = f(q)$. Then the family

$$(3) \qquad \mathscr{T}(D, p, q, P, Q) = \{f \in H_u(D): f(p) = P, \ f(q) = Q\},$$

normalized at two points, is of the form (1).

In order to solve extremal problems over such families $\mathscr{F}$, it is useful to know whether $\mathscr{F}$ is compact. Our first result (Theorem 1) is a characterization of the nonempty and compact families $\mathscr{F}$ in terms of the normalizing functionals ℓ_1 and ℓ_2 and the constants P and Q.

A function $f \in \mathscr{F}$ is said to be a *support point* of $\mathscr{F}$ if and only if $\Re\, L(f) = \sup_{\mathscr{F}} \Re\, L$, for some $L \in H'(D)$ that is not constant on $\mathscr{F}$. Geometrically, at a support point the family $\mathscr{F}$ has a supporting hyperplane. Theorems 2 and 3 (and 3') concern the mapping properties of support points for compact families $\mathscr{F}$. Applications to the families $\mathscr{S}(D, z_0)$ and $\mathscr{T}(D, p, q, P, Q)$ are contained in Theorems 4 and 5 (and 4' and 5').

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