ON NORMAL AND AUTOMORPHIC FUNCTIONS

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1. INTRODUCTION

Let Γ be a Fuchsian group, that is, a discontinuous group of Moebius transformations of the unit disk $D = \{|z| < 1\}$ onto itself. The points $z, z' \in D$ are called *equivalent* if there exists a mapping $\phi \in \Gamma$ such that $z' = \phi(z)$. A domain $F \subset D$ is called a *fundamental domain* of Γ if it does not contain two equivalent points and if every point in D is equivalent to some point in $\overline{F}$.

The function $f(z)$ will be called *character-automorphic* (with respect to Γ) if it is meromorphic in D and if

$$(1.1) \qquad f(\phi(z)) = v(\phi) f(z), \quad \text{where } |v(\phi)| = 1 \quad (z \in D,\ \phi \in \Gamma).$$

It follows from (1.1) that $v(\phi \circ \psi) = v(\phi)\, v(\psi)$ for $\phi, \psi \in \Gamma$, so that v is a character of Γ, and (1.1) is equivalent to $|f \circ \phi| = |f|$ $(\phi \in \Gamma)$. If $v(\phi) = 1$ for all $\phi \in \Gamma$, then $f(z)$ is *automorphic.*

We use the notation

$$(1.2) \qquad f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

for the spherical derivative. It is invariant under spherical rotations. The meromorphic function $f(z)$ $(z \in D)$ is called *normal* [8] if

$$(1.3) \qquad \sup_{z \in D} (1 - |z|^2) f^{\#}(z) = M < \infty .$$

This quantity is invariant under Moebius transformations $\phi(z)$ of D onto D. For character-automorphic functions, the supremum can therefore be restricted to any fundamental domain F. In particular, every bounded analytic function is normal. If $f(z)$ is analytic and normal, then [5], [13]

$$\log^+ |f(z)| \leq 2(\log^+ |f(0)| + M)(1 - |z|)^{-1} \qquad (z \in D).$$

We denote the non-Euclidean distance by $d(z_1, z_2)$ $(z_1, z_2 \in D)$ and the spherical distance by

$$(1.4) \qquad d^*(w_1, w_2) = \arctan \left| \frac{w_1 - w_2}{1 + \bar{w}_1 w_2} \right| \qquad (w_1, w_2 \in \hat{\mathbb{C}}).$$

If $f(z)$ is normal, then it follows from (1.3) by integration that

$$(1.5) \qquad d^*(f(z_1), f(z_2)) \leq M\, d(z_1, z_2) \qquad (z_1, z_2 \in D).$$

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