THE WHITEHEAD TORSION OF A FIBER-HOMOTOPY EQUIVALENCE

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1. INTRODUCTION AND STATEMENTS OF RESULTS

This paper is based on the observation that *if* $\xi = (E, p, B, F)$ *is a piecewise linear* (PL) *fiber bundle, then* p *induces a homomorphism* $p^*: \mathrm{Wh}\,\pi_1(B) \to \mathrm{Wh}\,\pi_1(E)$, *where* $\mathrm{Wh}\,\pi$ *denotes the Whitehead group of* π (see Proposition 2.3).

The definition of a PL fiber bundle is given in [1]. We can also completely determine the homomorphism p^* in many cases by using the results of [1].

We describe here the construction of the homomorphism p^*; for complete details we refer the reader to Section 2. Let $\tau_0 \in \mathrm{Wh}\,\pi_1(B)$ be arbitrary, and let $f: B' \to B$ be a PL homotopy equivalence such that $\pi(f) = \tau_0$, where $\tau(f)$ denotes the Whitehead torsion of f. Form the induced fiber space with total space

$$f^!(E) = \{(b', c) \in B' \times E \mid f(b') = p(e)\},$$

and notice that the map $g: f^!(E) \to E$ given by $g(b', e) = e$ is also a homotopy equivalence. Since f is PL, the space $f^!(E)$ inherits a PL structure in a natural way, and g has a Whitehead torsion $\tau(g)$. Define $p^* \tau_0 = \tau(g)$.

The following is our main result.

THEOREM A. *Let* $\xi_i = (E_i, p_i, B_i, F_i)$ $(i = 1, 2)$ *be* PL *fiber bundles with connected base and fiber, and let* $g: E_1 \to E_2$ *be a fiber-homotopy equivalence covering* $f: B_1 \to B_2$ *and inducing* $h: F_1 \to F_2$. *Then*

$$\tau(g) = p_2^* \tau(f) + \chi(B_2)\, j_{2*} \tau(h),$$

where $j_{2*}: \mathrm{Wh}\,\pi_1(F_2) \to \mathrm{Wh}\,\pi_1(E_2)$ *is induced by the inclusion* $j_2: F_2 \to E_2$.

We give the proof in Section 3. As a special case we obtain the following result, due to K. W. Kwun and R. H. Szczarba [7, Corollary 1.3].

COROLLARY B. *Let* $f: B_1 \to B_2$ *and* $h: E_1 \to E_2$ *be homotopy equivalences. Then*

$$\tau(f \times h) = \chi(F_2)\, k_{2*} \tau(f) + \chi(B_2)\, j_{2*} \tau(h),$$

where k_{2*} *is induced by the inclusion* $k_2: B_2 \to B_2 \times F_2$.

Proof. This follows from Theorem A if we set $g = f \times h$ and observe that the Product Theorem of [7] shows that $p_2^* \tau = \chi(F_2)\, k_{2*} \tau$ for each $\tau \in \mathrm{Wh}\,\pi_1(B_2)$, where $p_2: B_2 \times F_2 \to B_2$ is projection on the first factor.

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