A ROTUND REFLEXIVE SPACE HAVING A SUBSPACE OF CODIMENSION TWO WITH A DISCONTINUOUS METRIC PROJECTION

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If E is a strictly convex (rotund) and reflexive Banach space and L is a closed linear subspace of E, then L is a Chebyshev subspace of E; that is, corresponding to each point x in E there exists in L a unique point $P_L(x)$ that is nearest to x. The *metric projection* of E onto L is the mapping P_L. In [2], the authors constructed a strictly convex but nonreflexive Banach space possessing a linear subspace of codimension 2 whose metric projection is discontinuous. They conjectured that if L is a closed subspace of a strictly convex reflexive space, then P_L must be continuous. The conjecture is false even in spaces equivalent to a Hilbert space. An elegant counterexample was constructed by B. Kripke, and independently the present writer constructed a more complicated example, some features of which are more general. In both examples, the subspace L is of infinite codimension, and Ivan Singer, in a private communication, asked whether an example with a subspace L of codimension 2 could be constructed. Here we construct such an example by modifying our original method.

THEOREM. *There exist a strictly convex, reflexive, and separable real Banach space* E *and closed linear subspaces* L *and* M, *with* $L \subseteq M$, *having the properties*

(1) P_L *is discontinuous,*

(2) L *is of codimension* 2 *in* E, *and*

(3) M *is of codimension* 1 *in* E *and is a Hilbert space with respect to the norm of* E.

The construction of E depends upon a lemma asserting the existence of strictly convex norms with prescribed properties.

LEMMA. *Let* F *be a real linear space, and let* p_1 *and* p_2 *be two equivalent norms on* F *with respect to which* F *is separable. If* $p_1(x) \leq p_2(x)$ *for all* $x \in F$ *and the set* $\{x \in F: p_1(x) = p_2(x) = 1\}$ *contains no nondegenerate line segment, then there exists a strictly convex norm* p *on* F *with* $p_1(x) \leq p(x) \leq p_2(x)$ *for all* $x \in F$.

Proof. Throughout the proof, there will be a single topology on F, the norm topology determined by p_1 and p_2.

Suppose that y is a point of the open set $V = \{x \in F: p_1(x) < p_2(x)\}$. The first step in the proof is to show that there exists a norm p_y between p_1 and p_2 that is 'strictly convex near y'. Replacing y by a multiple, we shall suppose that $p_1(y) < 1 < p_2(y)$. Since F is p_2-separable, and by a well known result of J. A. Clarkson [1], there exists a strictly convex norm q on F that is equivalent to p_2. The norm p_y will be obtained as a modification of q.

Let $f \in F^*$ be a continuous linear functional on F that has p_2-norm equal to 1 and attains its norm at y: that is, such that $|f(x)| \leq p_2(x)$ for all $x \in F$, and $f(y) = p_2(y)$. Let

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