SYLVESTER'S PARTITION THEOREM, AND A RELATED RESULT

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For $k > 0$, let $\Pi_d(k)$ denote the set of partitions of k into distinct parts. For $\Pi \in \Pi_d(k)$, let $s(\Pi)$ be the number of sequences of consecutive integers in Π, and let $g(\Pi)$ be the number of gaps in Π. That is, let

$$g(\Pi) = s(\Pi) - 1$$

if the smallest part in Π is 1, while

$$g(\Pi) = s(\Pi)$$

if the smallest part in Π is greater than 1.

For $k > 0$ and $r \geq 0$, let $A(k, r)$ denote the number of partitions of k into odd parts (repetitions allowed) exactly r of which are distinct, $B(k, r)$ the number of $\Pi \in \Pi_d(k)$ with $s(\Pi) = r$, $C(k, r)$ the number of partitions of k into even parts (repetitions allowed) exactly r of which are distinct, and $D(k, r)$ the number of $\Pi \in \Pi_d(k)$ with $g(\Pi) = r$, and let

$$A(0, 0) = B(0, 0) = C(0, 0) = D(0, 0) = 1,$$

$$A(k, r) = B(k, r) = C(k, r) = D(k, r) = 0 \qquad \text{otherwise.}$$

We shall prove the following two results.

THEOREM 1. $B(k, r) = A(k, r)$ *for all* k *and* r.

THEOREM 2. $D(k, r) = C(k, r) + C(k - 1, r) + C(k - 3, r) + C(k - 6, r) + \cdots$ *for all* k *and* r.

Theorem 1 was proved arithmetically by J. J. Sylvester [4, Section 46]. Recently, G.E. Andrews [1, Section 2] gave a proof of Theorem 1 by means of generating functions. Our proofs also make use of generating functions; but they are more direct.

V. Ramamani and K. Venkatachaliengar [3, Section 2] have given a combinatorial proof of Theorem 1. A similar proof is available for Theorem 2.

For $k > 0$ and $r, n \geq 0$, let $B(k, r, n)$ denote the number of $\Pi \in \Pi_d(k)$ with $s(\Pi) = r$, and with no part greater than n, let

$$B(0, 0, n) = 1 \qquad \text{for } n \geq 0,$$

$$B(k, r, n) = 0 \qquad \text{otherwise,}$$

and let

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