ON INNER FUNCTIONS WITH H^p-DERIVATIVE

P. R. Ahern and D. N. Clark

In this paper, we consider the problem of determining the H^p-classes $(p > 0)$ to which the derivative ϕ' of an inner function ϕ in the unit disk U belongs. Various conditions sufficient for the relation $\phi' \in H^p$ have been established; see J. G. Caughran and A. L. Shields [5], M. R. Cullen [7], and H. A. Allen and C. L. Belna [2] for the case where ϕ is a singular inner function, and D. Protas [11] for the case where ϕ is a Blaschke product.

Part I is devoted to results about general inner functions. In Section 1, we give some relevant results on angular derivatives of bounded analytic functions. Applying these results to inner functions (Section 2), we show that the relation $\phi' \in H^{1/2}$ implies that ϕ is a Blaschke product. This answers a question raised in [5] and [7]; see also [2]. Another consequence is that $\phi' \in H^p$ if and only if $\phi'/s \in H^p$, where s is the singular part of ϕ. This has been proved before in a special case in [7]; the corresponding statement with H^p replaced by B^p is known to be false (see [2]). We conclude Part I with an application of our results to exceptional and omitted sets.

In the second part, we consider the case where ϕ is a Blaschke product with zeros $\{a_n\}$. Protas [11] (see also Caughran and Shields [6]) has shown that the condition

$$\sum (1 - |a_n|)^\delta < \infty \tag{1}$$

with $\delta = 1 - p$ is sufficient for $\phi' \in H^p$ if $1/2 < p < 1$. Using the results of Section 2, we show in Section 3 that condition (1), with $\delta = (1 - p)/p$, is necessary for $\phi' \in H^p$ $(1/2 < p < 1)$. This is apparently the first known necessary condition for the derivative of a Blaschke product to lie in H^p. We show by example that both these conditions represent the best possible values of δ. In fact, if the zeros a_n converge to a boundary point nontangentially, then $\delta = (1 - p)/p$ is precisely the right order of convergence of (1) (Section 4). Section 5 gives sufficient conditions for the relation $\phi' \in H^p$ in some other cases.

PART I. INNER FUNCTIONS

1. ANGULAR DERIVATIVES

We consider the class $\mathbb{B}$ of functions that are holomorphic and bounded (in modulus) by 1 in U. A function $f \in \mathbb{B}$ has a factorization of the form

$$f(z) = \left\{ \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \right\} \exp \left\{ - \int \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \right\}, \tag{2}$$

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