THE ABSOLUTE CONVERGENCE OF CERTAIN LACUNARY FOURIER SERIES

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Let G be a compact abelian group, and let Γ be its dual group. Suppose $E \subset \Gamma$ and f is a function on G. The function f is called an *E-function* if $\hat{f}(\gamma) = 0$ for all $\gamma \notin E$ ($\hat{f}$ is the Fourier transform of f). By A(G) we denote the space of functions whose transforms belong to $\ell^1(\Gamma)$, and $\|f\|_{A(G)}$ is defined to be $\|\hat{f}\|_{\ell^1(\Gamma)}$. For each set S(G) of functions defined on G, we denote by $S_E(G)$ the E-functions in S(G). A set $E \subset \Gamma$ is a *Sidon set* if $A_E(G) = C_E(G)$, where C(G) is the space of continuous functions on G. For $2 < p < \infty$, a set $E \subset \Gamma$ is a $\Lambda(p)$-*set* if $L^2_E(G) = L^p_E(G)$. A set $E \subset \Gamma$ is a Λ-*set* if it is a $\Lambda(p)$-set for all p and if in addition the inclusions $L^2_E(G) \to L^p_E(G)$ have norm at most $Cp^{1/2}$, where C depends only on the set E.

It is known that every Sidon set is a Λ-set [9, p. 128], and that there exist sets that are $\Lambda(p)$-sets for all p but are not Sidon sets [2, p. 803]. Actually, in the light of results in [1, p. 131], the sets constructed in [2] are not Λ-sets. It is therefore natural to ask whether there exist Λ-sets that are not Sidon sets. In general, this is an open question, but in certain torsion groups every Λ-set is also a Sidon set [6]. That Sidon sets are close to Λ-sets from a structural standpoint was shown in [1]. In this paper, we show that in an analytical sense they are also close. In particular, we construct a Banach space B(G) of functions on G such that $A(G) \hookrightarrow B(G) \hookrightarrow C(G)$ and such that $E \subset \Gamma$ is a Λ-set if and only if $A_E(G) = B_E(G)$. The construction of B(G) is motivated by the work in [3] and [4], and the connection between B(G) and Λ-sets is analogous to the connection between A. Figà-Talamanca's $A^p(G)$-spaces and $\Lambda(p)$-sets [5].

In Section 1 of this paper, we define two spaces K(G) and R(G) of functions on G that, (in the language of M. A. Rieffel [7]) are Banach modules. The space B(G) is then defined, and it turns out to be a realization of the Banach module tensor product $K(G) \otimes_{L^1(G)} R(G)$. In Section 2, we establish the connection between B(G) and Λ-sets.

1. DEFINITIONS AND PROPERTIES OF THE BASIC SPACES

For $f \in \bigcap_{2<p<\infty} L^p(G)$, let

$$\|f\|_\Lambda = \sup\{p^{-1/2}\|f\|_p \mid 2 < p < \infty\},$$

and let K(G) be the set of all functions f on G for which $\|f\|_\Lambda$ is finite. It is easy to verify that $\|\ \|_\Lambda$ is a norm on K(G) and that, endowed with this norm, K(G) becomes a two-sided Banach $L^1(G)$-module with respect to convolution. Next, we shall define a space R(G) of functions that is also a two-sided Banach $L^1(G)$-module

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