CONDITIONALLY CONVERGENT SERIES IN R^∞

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1. INTRODUCTION

Let A denote the infinite series $\sum_{k=1}^{\infty} a_k$, where $\{a_k\}_{k=1}^{\infty}$ is a sequence of elements of a topological vector space X. If p is a permutation of the positive integers, let A_p denote the series $\sum_{k=1}^{\infty} a_{p(k)}$, called a *rearrangement* of A. Let S_A denote the set of elements $s \in X$ such that some rearrangement of A converges to s. If A converges and S_A contains only one element, then A is said to *converge with invariant sum.* If A converges, but not every rearrangement of A converges, then A is said to *converge conditionally.* If A_p converges for every permutation p, then A is said to *converge unconditionally.*

In every linear topological space, unconditional convergence implies convergence with invariant sum. In a Euclidean space R^m, the converse is true. In fact, if A is a conditionally convergent series in R^m, then S_A is an affine subspace of R^m whose dimension is at least one. (In the case when m = 1, this result is of course a well-known theorem of Riemann (see [15, p. 419] or [1, Chapter 12]); proofs for the general case have been given by E. Steinitz [13] and others ([6], [14], [16], [17]).) In Section 2, we shall prove that the same statement holds for the countably-infinite product space R^∞ (with the product topology). Our treatment makes it easy to understand just how the dimension of S_A is determined, in either the finite- or infinite-dimensional case.

C. W. McArthur [11], using work of H. Hadwiger [9], showed that in every infinite-dimensional Banach space there is a conditionally convergent series that converges with invariant sum. His method yields the same result for every infinite-dimensional Fréchet space on which a continuous homogeneous norm can be defined. A Fréchet space has such a norm if and only if it does not contain a subspace isomorphic to R^∞ (see [2]).

We should like to mention the important result of A. Dvoretzky and C. A. Rogers [5], that in every infinite-dimensional Banach space there is a series that converges unconditionally but not absolutely. For other proofs of this, see [10], [12], and [7] or [8].

In Section 3, we consider another question about series in R^∞: Is it true that for every sequence $\{a_k\}_{k=1}^{\infty}$ in R^∞ such that $\lim_{k\to\infty} a_k = 0$, there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$, with each ε_k equal to +1 or -1, such that $\sum_{k=1}^{\infty} \varepsilon_k a_k$ converges? The answer is yes. The answer was known to be yes in the case of R^m [3] and no in the case of every infinite-dimensional Banach space [4, p. 157, Theorem 8].

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