THE HAUSDORFF METRIC AND CONVERGENCE IN MEASURE

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1. INTRODUCTION

Let m denote n-dimensional Lebesgue measure in R^n. If $\{C_k\}$ is a sequence of compact sets in R^n, convergent in the Hausdorff metric to a compact set C, the sequence $\{m(C \triangle C_k)\}$ may fail to converge to zero. For example, the unit disc in the plane is the Hausdorff limit of a sequence of finite sets. Equivalently, the sequence of characteristic functions $\{\chi_{C_k}\}$ may fail to converge in measure to the characteristic function of C. We characterize the sequences $\{C_k\}$ for which $\lim_{k\to\infty} m(C \triangle C_k) = 0$.

2. PRELIMINARIES

Let $B_\varepsilon(x)$ denote the closed ε-ball about a point x in R^n.

Definition. Let C be a compact set in R^n. The ε-*parallel body* $B_\varepsilon(C)$ is the compact set $\bigcup_{x\in C} B_\varepsilon(x)$. The ε-*annulus* $A_\varepsilon(C)$ is the compact set $B_\varepsilon(C) \setminus \operatorname{int} C$.

If C and K are compact subsets of R^n, the Hausdorff distance of C from K is

$$d(C, K) = \inf\{\varepsilon\colon B_\varepsilon(C) \supset K \text{ and } B_\varepsilon(K) \supset C\}.$$

If $\mathscr{A}$ denotes the collection of compact subsets of R^n, then $\langle \mathscr{A}, d\rangle$ is a complete metric space. Each closed and bounded subspace of $\langle \mathscr{A}, d\rangle$ is compact [1]. If $\{C_k\}$ is a sequence of compact sets such that $\lim_{k\to\infty} d(C_k, C) = 0$, then for each $\varepsilon > 0$, C_k is contained in $B_\varepsilon(C)$ for all sufficiently large integers k. Since $\lim_{\varepsilon\to 0+} m(B_\varepsilon(C)) = m(C)$, the assignment $C \to m(C)$ is an upper-semicontinuous function. In addition,

$$\lim_{k\to\infty} m(C \triangle C_k) = 0 \quad \text{if and only if} \quad \lim_{k\to\infty} m(C \setminus C_k) = 0.$$

3. RESULTS

To establish our characterization theorem, we shall use the following theorem of Dini. Let $\{f_k\}$ be a sequence of upper-semicontinuous nonnegative functions defined on a compact metric space Y. Suppose for each x in Y, the sequence $\{f_k(x)\}$ converges monotonically to zero. Then $\{f_k\}$ converges uniformly to the zero function on Y.

For $\ell = 1, 2, \cdots$ and for each compact set C in R^n, let $m_\ell(C)$ denote $m(B_{1/\ell}(C))$. Of course, the assignment $C \to m_\ell(C)$ determines an upper-semicontinuous function on $\langle \mathscr{A}, d\rangle$ for each ℓ.

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