A NEW DEFINITION FOR QUASISYMMETRIC FUNCTIONS

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1. INTRODUCTION

A continuous, strictly increasing function u mapping the line $(-\infty, \infty)$ onto itself is said to be ρ_0-*quasisymmetric* (or ρ_0 - QS for short) if $1 \leq \rho_0 < \infty$ and ρ_0 is the infimum of all ρ satisfying the condition

$$(1) \qquad 0 < \frac{1}{\rho} \leq \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq \rho < \infty$$

for all x and all $t > 0$. The number $\rho_0 = \rho(u)$ is called the *quasisymmetric dilatation* of u on $(-\infty, \infty)$.

L. V. Ahlfors and A. Beurling proved in 1956 [1] that an autohomomorphism u of the real line can be extended to a quasiconformal autohomomorphism f of the upper half-plane if and only if u is quasisymmetric. Furthermore, if K(f) is the quasiconformal dilatation of f, then

$$K(f) \geq 1 + (0.2284) \log \rho(u)$$

for each quasiconformal extension f of u, and there exists an extension $\hat{f}$ for which

$$K(\hat{f}) \leq [\rho(u)]^2 .$$

A good bound on $\rho(u)$ is therefore of great importance in any investigation of the quasiconformal extensions of u to the upper half-plane.

We begin by showing that (1) is really a generalized convexity-concavity condition, and that we can weaken the assumptions in the definition of quasisymmetry significantly without altering the class of such functions.

We then use this to prove that the class of quasisymmetric functions is closed under the formation of sums, appropriate products, compositions, and inverses. These properties have previously been established by means of similar properties of quasiconformal mappings, but our proofs use only the new definition of quasisymmetry and some elementary real analysis.

Finally, we use the new definition to obtain sharp bounds for the quasisymmetric dilatation of sums, products, compositions and inverses of quasisymmetric functions on $(0, \infty)$.

2. A NEW DEFINITION

As we pointed out in the introduction, a homeomorphism u of $(-\infty, \infty)$ onto itself can be extended to a QC map of the upper half-plane onto itself if and only if u is

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