A GENERALIZATION OF EPSTEIN ZETA FUNCTIONS

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In [1], we associated with certain polynomials a Dirichlet series that generalizes the Epstein zeta functions. In [2], we used various methods to study the analytic properties of the Dirichlet series. In this note, we obtain somewhat stronger results for certain special cases.

Let $F(X) = F(X_1, \cdots, X_n)$ be an integral form of degree δ such that the equation $F(x) = 0$ has no solutions in $\mathbb{R}^n$ except $x = 0$. We may assume that $F(x)$ is positive definite. It is obvious that for each k the equation $F(\gamma) = k$ has only finitely many solutions γ in $\mathbb{Z}^n$. Hence it makes sense to consider series of the type

$$\zeta(F, \alpha, s) = \sum_{\gamma \in \mathbb{Z}^n - \{0\}} F(\gamma)^{-s} e(\langle \alpha, \gamma \rangle),$$

where $s = \sigma + it$ is a complex number, $\alpha \in \mathbb{Z}^n$, the symbol $\langle\ ,\ \rangle$ indicates the standard inner product in $\mathbb{R}^n$, and $e(a) = \exp(2\pi i a)$ for $a \in \mathbb{R}$. If $F(x)$ is a quadratic form and $\alpha \in \mathbb{Z}^n$, then $\zeta(F, \alpha, s)$ is the well-known Epstein zeta function. The absolute convergence of the series for $\sigma > n/\delta$ in the general case and the analytic continuability for $\alpha \in \mathbb{Q}^n$ in certain special cases have been established in [1] and [2]. For $\alpha \in \mathbb{Q}^n$, we may apply C. L. Siegel's method [3] to continue the series analytically into the half-plane $\sigma > (n - 1)/\delta$ (see [2]).

In this paper, we shall prove the following result.

THEOREM. (a) *If* $\alpha \notin \mathbb{Z}^n$, *the function* $\zeta(F, \alpha, s)$ *can be continued analytically as an entire function of* s.

(b) *If* $\alpha \in \mathbb{Z}^n$, *the function* $\zeta(F, \alpha, s)$ *can be continued analytically as a meromorphic function of* s *with only a simple pole at* $s = n/\delta$; *the residue is*

$$\operatorname*{Res}_{s=n/\delta} \zeta(F, \alpha, s) = (2\pi)^{n/\delta}\, \Gamma(n/\delta)^{-1} \int_{\mathbb{R}^n} \exp(-2\pi F(x))\, dx.$$

Proof. Let us put $\xi(F, \alpha, s) = (2\pi)^{-s}\, \Gamma(s)\, \zeta(F, \alpha, s)$. By the Mellin transform, we get the integral representation

$$\xi(F, \alpha, s) = \int_0^\infty \sum_{\gamma \in \mathbb{Z}^n - \{0\}} \exp(-2\pi t F(\gamma))\, e(\langle \alpha, \gamma \rangle)\, t^{s-1}\, dt$$

$$= \int_0^\infty [\mathcal{O}(F, \alpha, it) - 1]\, t^{s-1}\, dt \qquad (s > n/\delta),$$

where, for $\tau \in H = \{z \in \mathbb{C}: \Im z > 0\}$,

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