A MINIMAL EXTENSION THAT IS NOT CONSERVATIVE

Robert G. Phillips

1. INTRODUCTION

Let L denote a countable first-order language that has relation symbols for addition, multiplication, and order, and that has 0 and 1 as constant symbols. P denotes Peano's axioms, either for the natural numbers or for the integers, formulated in L. When the context does not distinguish which set of axioms is being denoted by P, then the results or definitions involved are to be interpreted as being valid for either set.

When M is a model for P, then L_M will denote the extension of L obtained by adding the elements of $M - \{0, 1\}$ to L as constant symbols. It is to be understood that when formulas of L_M are interpreted in M, constant symbols will always denote themselves. A formula of L_M is called an *M-formula,* and a relation on M is called *M-definable* if it can be represented in M by an M-formula. If M^* is an extension of M, and R is an n-ary relation on M^*, then

$$\{(x_1, \cdots, x_n): x_1 \in M \wedge \cdots \wedge x_n \in M \wedge (x_1, \cdots, x_n) \in R\}$$

will be called the *restriction* of R to M.

Assume then that M models P and that M^* is a proper elementary extension of M with respect to L_M. If no proper elementary substructure of M^* properly extends M, then M^* is called a *minimal extension* of M. M^* is called a *conservative extension* of M if the restriction of each M^*-definable relation to M is also M-definable.

In [3], H. Gaifman formulated the concept of a minimal extension and proved that each model of P has a minimal extension. In [5], conservative extensions were introduced, and it was proved that each model of P has a conservative extension. It is the primary purpose of this paper to prove the following theorem.

THEOREM 1. *There exists a minimal extension of the standard model of* P *that is not a conservative extension.*

The theorem, and a related result, will be proved in the last section; this section is concluded with a brief account of the reasons that motivated the theorem.

First of all, it is clear that the concepts of minimal and conservative extension do not coincide. A conservative extension of a conservative extension of M is still a conservative extension of M, but it is obviously not a minimal extension of M. Nevertheless, there are certain similarities between the two concepts; a few of these are now listed:

(a) Gaifman's construction of minimal extension and that of conservative extension both depend upon the same basic principle. This principle is that for each unbounded M-definable subset X of M and for each M-definable relation Q on M, there exists an M-definable function f in 2^M such that

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