OBSTRUCTION THEORY IN THE STABLE RANGE

Donald W. Kahn

Obstruction theory is the branch of algebraic topology that studies the problems of extending or classifying maps, from a complex X to a space Y, one dimension at a time. Implicit in the fundamental work of H. Whitney [9], it was first codified officially by S. Eilenberg [2]. The most general formulation was given by P. Olum [7].

One basic concept in the classical obstruction theory is that of the obstruction set $\mathscr{O}^n(X, Y)$ (see for example p. 181 of [4]). This set consists of the set of all possible obstruction classes, that is, classes in $H^n(X; \Pi_{n-1}(Y))$ that arise as obstructions to the extension of maps from X^{n-1} to Y (here X^i means the i-skeleton of the complex X) to the n-skeleton. In general, the literature contains little information about obstruction sets. It seems that these sets have little interesting structure, without some special assumptions.

The purpose of the present paper, which concerns the theory rather than computations, is to show that the situation is radically different in the stable case. We show that when we stabilize with respect to suspension, the obstruction sets are filtered groups, so that the total obstruction set

$$\mathscr{O}^*_S(X, Y) = \sum_n \mathscr{O}^n_S(X, Y)$$

is a filtered, graded group. It is a subgroup of the total cohomology group

$$\sum_n H^n(X; \Pi^S_{n-1}(Y)),$$

where $\Pi^S_j(Z)$ means the jth-stable homotopy group of Z. Furthermore, if we consider this filtered, graded group for all suspensions of X,

$$\sum_j \mathscr{O}^*_S(\Sigma^j X, Y),$$

where j varies over the integers (including negative integers), and where Σ^j means, of course, the j-fold reduced suspension, we have the structure of a graded, filtered module over G_*, the stable homotopy ring of spheres. This module structure is compatible with the G_*-module structure on

$$\sum_j \{\Sigma^j X, Y\},$$

Received August 2, 1973.
This research was partially supported by NSF contract GP-34021.

Michigan Math. J. 21 (1974).