AN EXISTENCE THEOREM FOR PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

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1. INTRODUCTION

This paper is the culmination of a series of investigations by several authors. W. S. Loud should be credited with originating these studies. In [5] he proved the following theorem.

THEOREM 1.1. *Let* $g(x)$ *be an odd function of class* C^1. *If there exist an integer* n *and a positive number* δ *satisfying the condition*

$$(n+\delta)^2 \leq g'(x) \leq (n+1-\delta)^2,$$

then for each number E *the differential equation*

$$x'' + g(x) = E \cos t$$

has a unique 2π*-periodic solution, which is even and odd-harmonic.*

In [3], D. E. Leach partially generalized this theorem by showing that if $g(x)$ satisfies the inequality stated in Loud's theorem and if $g(0) = 0$, then for each continuous 2π-periodic function $e(t)$ the differential equation

$$x'' + g(x) = e(t)$$

has a unique 2π-periodic solution. In [2], A. C. Lazer and D. A. Sánchez considered the vector differential equation

$$x'' + \operatorname{grad} G(x) = p(t) = p(t+2\pi), \tag{1}$$

where $p \in C(R, R^n)$ and $G \in C^2(R^n, R)$. This equation represents the Newtonian equations of motion of a mechanical system subject to conservative internal forces and periodic external forces. Lazer and Sánchez were able to show that if there exist an integer N and numbers μ_N and μ_{N+1} such that

$$N^2 < \mu_N \leq \mu_{N+1} < (N+1)^2,$$

and if for all a in R^n

$$\mu_N I \leq \left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j}\right) \leq \mu_{N+1} I,$$

where I is the identity matrix, then (1) has at least one 2π-periodic solution. Later, Lazer [1] showed that under far less restrictive conditions, (1) has at most one 2π-periodic solution. In particular, Lazer's conditions assume the existence of two real,

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