PIERCING DISKS WITH TAME ARCS

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We give here a short proof of the following theorem of R. H. Bing.

THEOREM 0 (Bing [5]; see [3] for an earlier related theorem). *Suppose that* S *is a 2-sphere in* E^3, *that* A *is a rectilinear segment in* E^3, *and that* $\varepsilon > 0$. *Then there exists an* ε*-ambient isotopy* $H: E^3 \times I \to E^3 \times I$ *of* E^3, *fixed outside* $N(A \cap S, \varepsilon)$, *such that* $H_1(A) \cap S$ *is finite.* (*The map* $H_1: E^3 \to E^3$ *is the homeomorphism defined by the restriction* $H \mid E^3 \times \{1\}$.)

Bing's original proof of the Side Approximation Theorem (S.A.T.) [4] (see [7] for a new proof) was considerably complicated by the lack of a proof for Theorem 0; Bing first proved Theorem 0 by using both the Side Approximation Theorem and a number of its deep consequences. Our proof of Theorem 0 depends only on the existence of abundantly many *tame* arcs in S [2]; the existence of such tame arcs has recently been established independently of the S.A.T. (see [6, Sections 2 and 3]).

Definition. A compact metric space K is said to be a *regular compactum* if for each $\varepsilon > 0$ there exist a finite subset K_ε of K and a separation

$$K - K_\varepsilon = K_1 \cup \cdots \cup K_r \qquad \text{(separated)}$$

of $K - K_\varepsilon$ such that each of the sets $K_1, \cdots, K_r$ has diameter less than ε.

LEMMA. *If* K *is a regular compactum in* E^2 *and* A *is an arc in* E^2, *then for each* $\varepsilon > 0$ *there exists an* ε*-isotopy* $H: E^2 \times I \to E^2 \times I$ *of* E^2, *fixed outside* $N(A \cap K, \varepsilon)$, *such that* $H_1(A) \cap K$ *is finite.*

Proof. Since K is at most 1-dimensional, we may assume that $A \cap K$ is a 0-dimensional subset of Int A. Then $K \cap A$ is covered by the interiors of finitely many disjoint ε-subarcs $A_1, \cdots, A_k$ of Int A. There exist disjoint ε-disks $D_1, \cdots, D_k$ in E^2 such that for each i the intersection $D_1 \cap A = A_i$ is a spanning arc of D_i. It clearly suffices to show that Bd A_i bounds a spanning arc B_i of D_i such that $B_i \cap K$ is finite.

Since K is a regular compactum, it follows from [8, (3.2) Theorem, p. 35] that there are finitely many disks $E_1, \cdots, E_n$ in Int D_i whose interiors cover $K \cap A_i$ and such that, for each j, the set (Bd E_j) $\cap$ K is finite. Clearly, there is an arc B_i bounded by Bd A_i in the set

$$\left[A_i - \bigcup_{j=1}^{n} \operatorname{Int} E_j \right] \cup \left[\bigcup_{j=1}^{n} \operatorname{Bd} E_j \right].$$

It is equally clear that this B_i must satisfy the requirements of the preceding paragraph. This completes the proof of the lemma.

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