THE VOLUME OF A SMALL GEODESIC BALL OF A RIEMANNIAN MANIFOLD

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1. INTRODUCTION

Let M be an analytic Riemannian manifold. For $m \in M$, let $V_m(r)$ denote the volume of a geodesic ball centered at m with radius r. Then $V_m(r)$ can be expanded in a power series in r. In this note we compute the first five terms in this expansion. Our computation shows, for example, that if the Ricci scalar curvature of M is positive, then for small r, $V_m(r)$ is less than the corresponding function for Euclidean space. More generally if M is a C^∞ Riemannian manifold, we can compute the Taylor expansion of $V_m(r)$, although it may not converge.

In order to compute the Taylor expansion of $V_m(r)$, it is necessary to discuss general power series expansions of tensor fields in normal coordinates. In Section 2, we present a method for computing such expansions in modern notation. The coefficients of the power series expansions are polynomials in the covariant derivatives of the tensor fields and the curvature tensor.

Normal coordinate power series expressed in terms of the curvature operator occur implicitly in the classical literature of differential geometry, for example in books and papers by E. Cartan [5], L. P. Eisenhart [7], T. Y. Thomas [15], O. Veblen [16], and Veblen and Thomas [17]. Also, explicit formulas are given by A. Z. Petrov [11].

I know of several uses for power series expansions in normal coordinates, and probably there are many more. For example, such expansions have been used in the theory of harmonic spaces (see [12], for example) and in determining the asymptotic expansion for $\sum e^{-\lambda_i t}$, where the λ_i are the eigenvalues of the Laplacian of a compact Riemannian manifold.

Recently, P. Gilkey [8] also used them to give an analytic proof of the index theorem.

In Section 3, we use the results of Section 2 to compute the power series expansion of $V_m(r)$, and we derive several consequences of this expansion. Also, using the method of [1], we compute $V_m(r)$ explicitly for symmetric spaces of rank 1.

The coefficient of r^{n+4} in the expansion of $V_m(r)$ is especially interesting. (Here $n = \dim M$.) It is a quadratic invariant of O(n). In Section 4, we compare it with other quadratic invariants arising from geometrical considerations. Notable among these are the conformal and spectral quadratic invariants and the 4-dimensional Gauss-Bonnet integrand. We discuss the linear independence among these and the quadratic invariant derived from $V_m(r)$ described above.

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