LOCAL COMPLEMENTS TO THE HAUSDORFF-YOUNG THEOREM

John J. F. Fournier

1. INTRODUCTION

Let G be an infinite, locally compact, Abelian group with dual group Γ. For $1 \leq p \leq \infty$, denote by $L^p(G)$ the usual Lebesgue space relative to the Haar measure on G; define $L^p(\Gamma)$ similarly. The Hausdorff-Young theorem [11, Vol. II, p. 227] states that if $1 < p < 2$, then with every function f in $L^p(G)$ there is associated a function $\hat{f}$ in $L^{p'}(\Gamma)$, where p' is the index conjugate to p; the mapping $f \mapsto \hat{f}$ is a bounded linear operator from $L^p(G)$ to $L^{p'}(\Gamma)$, and $\hat{f}$ is the usual Fourier transform of f whenever $f \in L^1(G) \cap L^p(G)$. Accordingly, for $1 \leq p \leq 2$, let

$$FL^p = \{g \in L^{p'}(\Gamma): g = \hat{f} \text{ for some } f \text{ in } L^p(G)\}.$$

For measurable sets $E \subset \Gamma$, denote by $FL^p \mid E$ the set of all functions on E that are restrictions to E of functions in FL^p. Clearly, $FL^p \mid E \subset L^{p'}(E)$. This paper deals with the possibility that $FL^p \mid E \subset L^q(E)$ for some $q \neq p'$.

If E is either finite or locally null [11, Vol. I, p. 124], then all of the spaces $L^q(E)$ for $q < \infty$ coincide. To avoid such trivialities, we assume for the rest of this paper that the set E is infinite and not locally null. In two cases, it follows from the Hausdorff-Young theorem that $FL^p \mid E \subset L^q(E)$ for some $q \neq p'$. First, if Γ is discrete, then

$$FL^p \mid E \subset L^{p'}(E) \subset L^q(E) \quad \text{for all } q \geq p'.$$

Second, if the Haar measure $|E|$ of E is finite, then

$$FL^p \mid E \subset L^{p'}(E) \subset L^q(E) \quad \text{for all } q \leq p'.$$

Thus the interest lies in the remaining cases:

(i) Γ is not discrete, and $q > p'$;

(ii) $|E| = \infty$ and $q < p'$.

The following three theorems constitute the main results of this paper.

THEOREM 1. *If* Γ *is not discrete and* E *is not locally null, then*

$$FL^p \mid E \not\subset \bigcup_{q > p'} L^q(E).$$

THEOREM 2. (a) *If* Γ *is not discrete, then* $FL^p \not\subset \bigcup_{q > p'} L^q(\Gamma)$.

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