THE SEGMENTAL VARIATION OF HOLOMORPHIC FUNCTIONS

Walter Rudin

E. Lindelöf and P. Montel proved the following theorems about the class H^∞ of all bounded holomorphic functions in the open unit disc U:

(a) If $f \in H^\infty$ and f has a limit, say L, along some arc in U that terminates at the point 1, then the radial limit of f exists at the point 1 and equals L.

(b) If $f \in H^\infty$ and f has a radial limit at 1, then f actually has a nontangential limit at 1.

The union of these statements is often called the *sectorial-limit theorem*. For a proof we refer to [1, Theorem 6.7].

These theorems suggest two questions, obtained by replacing the property of having a limit by the stronger one of having finite total variation:

(A) If $f \in H^\infty$ and f has finite total variation on some arc in U with one end-point at 1, does it follow that f has finite total variation on the radius $[0, 1)$?

(B) If $f \in H^\infty$ and f has finite total variation on $[0, 1)$ must the same be true on other line segments in U that end at 1?

An affirmative answer to (A) would lead to a quick proof that every $f \in H^\infty$ has finite total variation on some radius. (This possibility was not ruled out in [2].) However, we shall see that both (A) and (B) have negative answers, even if H^∞ is replaced by the disc algebra A, that is, by the class of all continuous functions on the closed unit disc $\overline{U}$ that are holomorphic in U.

To state the result concisely, we associate with each $\alpha \in (-\pi/2, \pi/2)$ the segment

$$I(\alpha) = \{1 - te^{i\alpha}: 0 < t < \cos \alpha\}, \tag{1}$$

and we define $V(f, \alpha)$ to be the total variation of any $f \in H^\infty$ on $I(\alpha)$:

$$V(f, \alpha) = \int_0^{\cos \alpha} |f'(1 - te^{i\alpha})|\, dt. \tag{2}$$

Note that one end-point of $I(\alpha)$ is 1 and that the other lies in U. Also, $I(\alpha)$ lies above $I(\beta)$ if and only if $\alpha < \beta$.

THEOREM. *To every* $\beta \in (-\pi/2, \pi/2)$ *correspond functions* f, g, h *in the disc algebra* A *such that*

(i) $V(f, \alpha) < \infty$ *if and only if* $\alpha < \beta$,

(ii) $V(g, \alpha) < \infty$ *if and only if* $\alpha \leq \beta$,

(iii) $V(h, \alpha) < \infty$ *if and only if* $\alpha = \beta$.

Received February 14, 1973.
The author was partially supported by NSF Grant GP-24182.

Michigan Math. J. 20 (1973).