OVERRINGS AND DIVISORIAL IDEALS OF RINGS OF THE FORM D + M

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1. INTRODUCTION

Let V be a valuation ring of the form K + M, where K is a field and M is the maximal ideal of V. If D is a subring of K, we denote by D_1 the subring D + M of V. Domains of this kind arise frequently in the literature, especially in connection with the construction of examples; see, for example, [35, p. 670], [46, p. 328], [38], [45, p. 604], [32, Chapter 4], [24, Section 5], [13, p. 252], [14, p. 500], [5, p. 305], [11, p. 280], [18, Section 4], and [6]. We compile in Theorem 2.1 the results that appear in [15] concerning domains of the form D + M. In Section 3 (Theorem 3.1), we determine the set of overrings of D_1. The theorem leads to numerous results involving special conditions on the set of overrings of domains that have been considered in the literature; these include GQR-domains [28], QQR-domains [18], domains for which each overring is an ideal transform [6], domains satisfying the transform formula for ideals [21], and domains for which the set of overrings is closed under addition [22].

In Section 4, we determine the set of divisorial ideals of D_1, and we investigate the condition, introduced by W. Heinzer in [27], that each ideal of D_1 be divisorial. We conclude, in Section 5, by deriving an expression for the dimension of the polynomial ring $D_1[X_1, \cdots, X_n]$ in terms of the dimension of $D[X_1, \cdots, X_n]$; but the main contribution of Section 5 is the information concerning realization of a sequence $\{n_0, n_1, n_2, \cdots\}$ in the form $\{\dim R, \dim R[X_1], \dim R[X_1, X_2], \cdots\}$. Our results in Section 5 depend strongly on a theorem of J. T. Arnold in [1].

Overall, our results show again what previous results concerning D_1 have indicated, namely, that the structure of D_1 reflects properties of the valuation ring V and properties of K as a ring extension of D. The richness of properties that can be realized by a construction of this type is usually due to the freedom involved in the choice of D and K.

2. A SUMMARY OF SOME KNOWN RESULTS

Because we shall need them frequently, we list in Theorem 2.1 some known results concerning domains of the form D + M. Detailed proofs for these results can be found in [15, Appendix 2].

2.1. THEOREM. *Let* V *be a nontrivial valuation ring with quotient field* L, *and assume that* V *is of the form* K + M, *where* K *is a field and* M *is the maximal ideal of* V. *Let* D *be a domain with identity that is a proper subring of* K, *and let* $D_1 = D + M$.

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