HAUSDORFF DIMENSION AND APPROXIMATION OF SMOOTH FUNCTIONS

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We prove two theorems about differentiable transformations of sets of a specified Hausdorff dimension; the first theorem concerns the dimension of certain intersections, and it complements a theorem of J. M. Marstrand [7, Theorem III, p. 275], while the second extends results of J.-P. Kahane and R. Salem (see [2, Chapter 15], [3, Chapter 8], and [9]) on the behavior at infinity of certain Fourier-Stieltjes transforms. In both cases, we demonstrate the existence of an extremal set in a specified class, by a combination of probability theory and quantitative approximation theory.

This paragraph contains the estimates necessary for approximation; the field is largely the creation of A. N. Kolmogorov, and the material is found in [6, Chapter 10] under the name "entropy". Let S be some collection of real-valued functions on an interval $[a, b]$ whose derivatives of order $0, 1, \cdots, k$ are uniformly bounded on $[a, b]$, for a positive integer k. For each $\varepsilon > 0$, we choose a set $S^* \subseteq S$ so that for each f in S there is an f^* in S with $|f(x) - f^*(x)| \leq \varepsilon$ throughout $a \leq x \leq b$. For small $\varepsilon > 0$, we can choose S^* so that its size $|S^*|$ satisfies an inequality of the form $\log |S^*| \leq C\varepsilon^{-1/k}$, where C depends on S but not on ε. This estimate is valid for fractional values of k, for which the analogue of C^k is defined as follows. We let $k_1 = [k]$, and we admit classes S bounded above in $C^{k_1}[a, b]$, imposing a Lipschitz condition with exponent $\alpha = k - k_1$ on the k_1-st derivative:

$$|f^{(k_1)}(x) - f^{(n_1)}(y)| \leq C\,|x - y|^\alpha \qquad (f \in S,\ a \leq x,\ y \leq b).$$

Before turning to the theorems, we point out two technical details that should be of interest to specialists. The first theorem involves not only probability and approximation, but also a function-space argument borrowed from Fourier analysis. To prove the second theorem, we need a somewhat difficult estimate of exponential integrals; but we use only elementary inequalities from probability, in contrast with [2, Chapter 15].

1. To explain the significance of the first theorem, we denote by F a closed linear set, and by μ a probability measure in F satisfying a Lipschitz condition $\mu(a, a + h) \leq C_\beta h^\beta$ for each interval $(a, a + h)$ and each exponent $\beta < \alpha \leq 1$. Then the planar set $F \times F$ carries the measure $\mu \times \mu$, which fulfills a Lipschitz condition for each exponent $2\beta < 2\alpha$. We can apply the method of Marstrand [7, Lemmas 10 to 19] to the set $F \times F$ (using $\mu \times \mu$ in place of $\Lambda^{2\alpha}$) to prove the following result: There exists a line $y = mx + b$ $(m \neq 1)$ whose intersection with F has dimension at least $2\alpha - 1$. This means that there is an affine map $T \neq 1$ of the line — hence an infinitely differentiable map with exactly one fixed point — such that $T(F) \cap F$ has dimension at least $2\alpha - 1$. In Theorem 1, we prove that the constant $2\alpha - 1$ is best

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