APPLICATIONS OF MAPPING THEOREMS TO SCHWARTZ SPACES AND PROJECTIONS

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1. INTRODUCTION

S. A. Saxon [13] has shown that if E is a locally convex linear topological space (henceforth: an LCLT space) that is nuclear and X is an infinite-dimensional Banach space, then there exists an embedding of E into some power X^I of X. Saxon's theorem improves a result of A. Grothendieck (see [14, p. 102]) who had proved the theorem in the cases where $X = \ell_p$ for some p $(1 \leq p \leq \infty)$. This raises the question as to what LCLT spaces are likewise embeddable in a sufficiently high power of each infinite-dimensional Banach space.

In the first part of this paper, we show that each LCLT space E that is embeddable in a sufficiently high power of every infinite-dimensional Banach space is a Schwartz space. Actually, we do not need the full strength of embeddability in powers of every Banach space. In fact, embeddability in powers of any of a large number of pairs of Banach spaces (see Proposition 2 for a listing of the pairs) is sufficient to imply that E is a Schwartz space. Since c_0 occurs as one of the members of such a pair, and since every Schwartz space is embeddable in a sufficiently high power of c_0 (see D. J. Randtke [10]), our result may be near to characterizing Schwartz spaces. It remains open, for example, to determine whether for every p $(1 \leq p < \infty)$ every Schwartz space is embeddable in some power of ℓ_p.

The result on Schwartz spaces depends on Proposition 1, which concerns the factorization of mappings of subspaces of a product space into a normed linear space. Proposition 3 concerns normed linear subspaces of products of LCLT spaces, and it leads to some results on projections of LCLT spaces onto Banach spaces (see the next paragraph for a sketch, Section 3 for details).

It is often desirable to determine whether a subspace of a normed linear space X is the image of a continuous projection on X. This is important because the identification of such subspaces usually provides structural information about the space. A. Sobczyk's classical result in this direction states that if X is a separable normed linear space and Y is a linear subspace of X topologically isomorphic to c_0, then there exists a continuous projection of X onto Y (see [15]). A more recent result concerning projections, proved by H. P. Rosenthal in [11], asserts that if X and Y are closed, totally incomparable linear subspaces of a Banach space, then X + Y is closed. It then follows that X is complemented in X + Y.

The problem of determining the subspaces of an LCLT space that are images of continuous projections is more difficult. However, if the subspace in question is a Banach space, some interesting results can be obtained. We show that the results of Sobczyk and Rosenthal hold in a more general setting, namely, where the underlying space is an LCLT space. For instance, we prove that if E is a separable LCLT space and F is a linear subspace of E topologically isomorphic to c_0, then there exists a continuous projection of E onto F (Theorem 2). We also prove that if F is a Banach space such that F is complemented in every Banach space that

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