# AN INVARIANT-SUBSPACE THEOREM

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## 1. INTRODUCTION

Throughout this paper, $\mathscr{H}$ will denote an infinite-dimensional, separable, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$. Recall from [5] that an operator S in $\mathscr{L}(\mathscr{H})$ is *quasitriangular* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of orthogonal projections of finite rank on $\mathscr{H}$, converging strongly to 1, such that

$$\|P_n S P_n - SP_n\| \to 0 . \tag{1}$$

Such a sequence $\{P_n\}$ satisfying (1) will be said to *implement* the quasitriangularity of S.

It is not known whether every quasitriangular operator has a nontrivial invariant subspace. The problem is important, because all quasinilpotent operators and all operators with compact imaginary part (acting on a separable space) are quasitriangular [4], [5]. Some invariant-subspace theorems concerning quasitriangular operators have been proved [1], [2], [3]. Perhaps the most interesting of these theorems is the result of W. B. Arveson and J. Feldman in [2], which asserts that if S is a quasitriangular operator in $\mathscr{L}(\mathscr{H})$ and there exists a sequence of polynomials $p_n(S)$ that converges in the norm topology to a nonzero compact operator, then S has a nontrivial invariant subspace.

The principal purpose of this note is to prove a generalization of this fundamental theorem. In order to state our main result, we introduce the following terminology and notation. If $T \in \mathscr{L}(\mathscr{H})$, we denote by $\mathscr{P}(T)$ the subalgebra of $\mathscr{L}(\mathscr{H})$ that is the uniform closure of the set of all polynomials in T. Furthermore, we denote by $\mathscr{R}(T)$ the subalgebra of $\mathscr{L}(\mathscr{H})$ that is the uniform closure of the set of all rational functions of T (a rational function of T is an operator of the form $p(T)[q(T)]^{-1}$, where p and q are polynomials). It is well known that $\mathscr{R}(T)$ coincides with the uniform closure of the algebra of all analytic functions of T. For this reason, a subspace $\mathscr{M}$ of $\mathscr{H}$ is called an *analytically invariant* subspace for T (see [6]) if it is invariant under every operator in $\mathscr{R}(T)$. In general, an invariant subspace $\mathscr{M}$ of an operator T need not be an analytically invariant subspace for T. In fact, it was shown in Lemma 2.2 of [6] that an invariant subspace $\mathscr{M}$ of T is analytically invariant for T if and only if the spectrum of $T \mid \mathscr{M}$ is contained in the spectrum of T.

Our principal result is the following theorem.

THEOREM 1.1. *Let* S *be a quasitriangular operator in* $\mathscr{L}(\mathscr{H})$, *and suppose that the algebra* $\mathscr{R}(S)$ *contains a nonzero compact operator. Then* S *has a nontrivial analytically invariant subspace.*

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