THE GREEN FUNCTION OF DOMAINS CONTAINING A FIXED ELLIPSE

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INTRODUCTION AND SUMMARY

Recently, E. Złotkiewicz and the present author [4] showed that domains of hyperbolic type have a property of "uniform local convexity." More precisely, if Ω is a domain of hyperbolic type, then any two points $w_1, w_2 \in \Omega$ whose hyperbolic distance $h(w_1, w_2; \Omega)$ with respect to Ω is less than $\tanh^{-1}(1/\sqrt{2})$ can be joined in Ω by a segment $[w_1, w_2]$. The constant $\tanh^{-1}(1/\sqrt{2})$ is the best possible.

The natural question arises whether the segment $[w_1, w_2]$ can be replaced by a larger set, after a suitable diminution of hyperbolic distance. In fact, if $0 < r < 1/\sqrt{2}$ and $h(w_1, w_2; \Omega) = \tanh^{-1} r$, then Ω contains an open ellipse with foci w_1 and w_2 and with eccentricity $\varepsilon(r) = 2r\sqrt{1 - r^2}$ (Theorem 3). In order to prove Theorem 3, we first solve an extremal problem involving the Green function $g(0, 1; \Omega)$ of domains Ω containing a fixed, maximal ellipse E with foci 0 and 1 (Theorem 1). Next, we consider a related problem for ring domains (Theorem 2). The well-known ring domain of A. Mori turns out to be extremal in this case. As corollaries of Theorem 3, estimates for the Green function $g(w_1, w_2; \Omega)$ are obtained under the assumption that Ω contains a fixed maximal ellipse with foci w_1 and w_2 (Theorem 4). As a consequence of Theorem 3 we also obtain a result that extends to arbitrary univalent majorants a theorem recently proved by Z. Lewandowski and J. Stankiewicz [6] for starlike majorants (Theorem 5).

1. TWO EXTREMAL PROBLEMS IN CONFORMAL MAPPING

We shall be concerned with the maximal value of the Green function $g(b, c; \Omega)$ for the class of simply connected domains Ω in the finite plane $\mathbb{C}$, each Ω containing a fixed ellipse E with foci b and c. Obviously, we may assume that $b = 0$ and $c = 1$, and that some boundary points of Ω actually lie on the boundary ∂E of E. We show that the extremal domain is the finite plane minus a ray on the prolongation of the minor axis of E.

THEOREM 1. *Let $\{\Omega\}$ be the class of simply connected domains Ω in the finite plane $\mathbb{C}$, each Ω containing the open ellipse E with foci 0 and 1 and with eccentricity ε. Let us also assume that the intersection $(\mathbb{C} \setminus \Omega) \cap \partial E$ is not empty. Then the Green function $g(0, 1; \Omega)$ is a maximum for $\Omega = \Omega_0 = \mathbb{C} \setminus \ell_0$, where ℓ_0 is one of the two vertical rays that lie outside of E and join the ends of the minor axis of E to the point at infinity. Moreover,*

$$(1.1) \qquad g(0, 1; \Omega_0) = -\frac{1}{2}\log\frac{1}{2}(1 - \sqrt{1 - \varepsilon^2}) = -\log\frac{1}{2}\sqrt{2 - \sqrt{4 - a^{-2}}},$$

where $2a = 1/\varepsilon$ is the major axis of E.

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