THE EXISTENCE OF AN EVERYWHERE DENSE INDEPENDENT SET

Frederick Bagemihl

Suppose that to every real number x there corresponds a set S(x) of real numbers such that $x \notin S(x)$. Two real numbers x and y are said to be *independent* provided that $x \notin S(y)$ and $y \notin S(x)$. A set of real numbers is said to be independent provided every pair of numbers in this set is independent.

P. Erdös has shown [2, Theorem 6] that if S(x) is nowhere dense for every x, then there exists an enumerable independent set. I shall prove the following stronger theorem.

THEOREM 1. *If* S(x) *is nowhere dense for every* x, *then there exists an everywhere dense independent set.*

It is interesting to note that if instead of assuming that each set S(x) is nowhere dense, one makes the assumption that no point x is a limit point of the corresponding set S(x), then there does not necessarily exist an everywhere dense independent set [1, Theorem 2].

The proof of the theorem uses facts about Baire category that can be found in [3]. Let $S^{-1}(y)$ denote the set of real numbers x for which $y \in S(x)$.

LEMMA. *Let* M *be a set that is everywhere of second category. Then there exists a residual set* R *such that, for every* $y \in R$, *the set* $M - S^{-1}(y)$ *is everywhere of second category.*

Assume that the conclusion of the lemma is false. Then there exists a set T of second category such that, for every $y \in T$, the set $M - S^{-1}(y)$ is not everywhere of second category. Hence, to every $y \in T$ there corresponds an open interval with rational endpoints such that the intersection of this interval with $M - S^{-1}(y)$ is a set of first category. Since there are only enumerably many such rational intervals and T is of second category, there exists a subset T_0 of T of second category such that to every $y \in T_0$ there corresponds the same rational interval J. There exists a subset T_1 of T_0 that is everywhere of second category in some open interval K. Hence, there exists an enumerable subset T_2 of T_1 that is everywhere dense in K. The set

$$J \cap \bigcup_{y \in T_2} [M - S^{-1}(y)]$$

is of first category, while $J \cap M$ is of second category. Therefore there exists an $x \in J \cap M$ for which $T_2 \subseteq S(x)$, which contradicts the fact that S(x) is nowhere dense. Thus our assumption is untenable, and the lemma is true.

Now to prove Theorem 1, let $\{J_1, J_2, \cdots, J_n, \cdots\}$ be the set of rational intervals. Denote the set of real numbers by M_1. According to the lemma, there exists a residual set R_1 such that for every $y \in R_1$, the set $M_1 - S^{-1}(y)$ is everywhere of

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